investigative efforts have been devoted to improving ASW sensors as well as the TMA techniques applied to the data from these sensors. These efforts have been primarily in a submarine context, but they have also become important to surface and airborne ASW and to anti-surface warfare (ASUW) by all platforms.

TMA is almost synonymous with the term "tracking" and the objectives are identical. In tracking, updates of the estimate of target position and velocity alternate between a motion update based on prior target motion assumptions and an information update based on new information from observations. Both updates are probabilistic. This is akin to the alternation between updates for motion and negative information (unsuccessful search) described in Chapter 8 in search for moving targets. Tracking, however, involves positive as well as negative information (as does search in more advanced treatments). Tracking algorithms typically process numerous observations via sophisticated forms of statistical regression, notably Kalman filtering, which are beyond the scope of this text.

A vessel's track will be said to be **linear** if its course and speed are constant. *Throughout this chapter* it is assumed that the target is on a linear track. The focus is on how much TMA can be done with two, three, or four bearings.

Some notation is defined in 1101. Section 1102 examines what can be obtained from bearings if own ship is also on a linear track. In that circumstance, a complete TMA solution is not possible no matter how much bearing information is available, but the direction of relative motion can be determined from three or more bearings. The important **Ekelund ranging** method, based on bearing rates before and after a turn by own ship is presented in 1103; these two rates may be approximated by use of four bearings. Section 1104 gives the time correction method for improving the Ekelund range estimate, by finding times at which the range estimation is insensitive to target speed in the line of sight.

Section 1105 develops the techniques of **Spiess** TMA. Given bearings at three times, the locus of target positions at a chosen fourth time is a computable straight line called the **Spiess line** at that time. Also discussed are connections among time correction, Spiess lines, target tracks consistent with three bearings, and the parabolic envelope to the latter two sets of lines.

1101 Notation

The following notation conventions will apply throughout the chapter. As elsewhere, target speed is u and target true course, i.e., relative to 000, is c. Time instants will be denoted $t_0, t_1, \dots$. At these respective times the bearings from own ship to target will be $b_0, b_1, \dots$, and the ranges will be $r_0, r_1, \dots$. The following definitions are made for $i, j = 0, 1, \dots$ (a double subscript ij refers to a change taking place from time t_i to t_j):

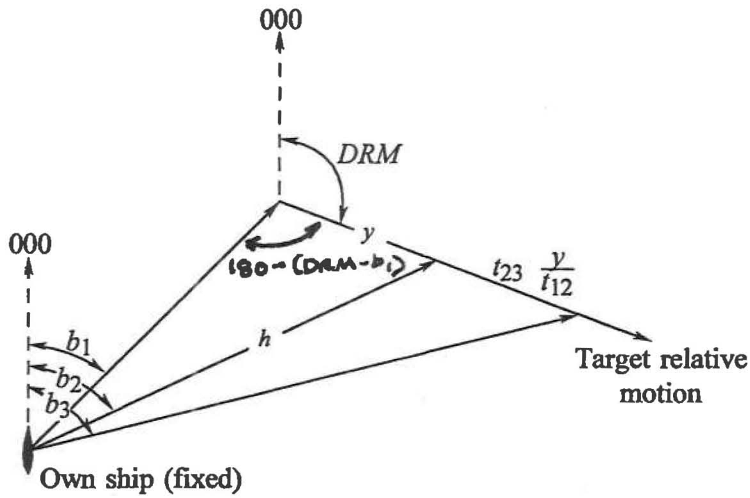
a. $t_{ij} = t_j - t_i$,
b. $b_{ij} = b_j - b_i$,
c. DOA_{ij} = distance own ship moves from t_i to t_j perpendicular to bearing b_i, i.e. *across* the b_i line of sight (if the motion is to the left of b_i, the distance is negative),
d. DOI_{ij} = distance own ship moves from t_i to t_j in the direction b_i, i.e. *in* the b_i line of sight (if the motion opens own ship from the target, the distance is negative),

e. DTA_{ij} and DTI_{ij} are the corresponding distances for *target* motion.

The above definition of b_{ij} must be interpreted in the arithmetic of the nautical compass. Specifically, b_{ij} is the change in bearing from time t_i to time t_j, measured as an angle between -180° and 180°, a clockwise change being positive. E.g., if $b_i = 010$ and $b_j = 355$, then $b_{ij} = -15°$. (See problem 22.)

The **direction of relative motion** (*DRM*) is defined as the direction of the vector of target motion relative to own ship, when own track is linear – see Figure 11.1. This should not be confused with target relative course, which is the angle measured clockwise from own course to target course.

FIGURE 11.1. DERIVATION OF *DRM*



1102 Own Ship on Linear Track

In this section it is assumed that own ship's track is linear, as has already been assumed for the target. The theorems below identify certain features of target motion that can and cannot be determined from bearings only in this case.

Two tracks are said to be on a **collision course** if at some instant of time they are at the same point – this could occur in past or future time.

Theorem 11.1. If the two tracks are on a collision course, then all bearings are equal.

Proof: Consider the track of the target plotted relative to own ship. Bearings are the same in the relative and true plots. If the two true tracks are on a collision course, then the relative track will pass through own ship. Each bearing runs from own ship to target in the relative plot; thus all bearings coincide.

Theorem 11.2. If the bearings at two distinct times are equal, then all bearings will be equal and the two tracks are on a collision course or are parallel.

Proof: In the plot relative to own ship, both bearing lines coincide at own ship (which is fixed), so they coincide entirely. Hence the relative motion lies within the bearing line. If target position relative to own ship is constant, then the two tracks are parallel. Otherwise, the constant relative velocity must take the target through own position, i.e., the two tracks collide, and by Theorem 11.1 all bearings are equal.

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Theorem 11.3. If t_1, t_2, and t_3 are distinct times, then

$$t_{23}\cot(DRM - b_1) = t_{13}\cot b_{13} - t_{12}\cot b_{12}. \tag{11-1}$$

In particular, if the times are uniformly spaced, i.e., $t_{23} = t_{12}$, then

assume $t_{13} = 2 \cdot t_{23}$

$$\cot(DRM - b_1) = 2\cot b_{13} - \cot b_{12}.$$

$\cot(DRM - b_1) = 2\cot b_{13} - \cot b$

$DRM = 90°$

Also, defining b_0 as the bearing at closest approach of the two tracks,

$\cot(90° - b_1) = \sin b_1$

$$t_{23}\tan b_{01} = t_{13}\cot b_{13} - t_{12}\cot b_{12}. \tag{11-2}$$

$t_{23} \cdot \tan b_{01} = t_{13} \cdot \cot b_{13} - t_{12} \cot b$

Equation (11-1) is solved for *DRM*, by computing the two values, which differ by 180°, of

$$b_1 + \operatorname{arccot}\left(\frac{t_{13}\cot b_{13} - t_{12}\cot b_{12}}{t_{23}}\right), \tag{11-3}$$

and adding or subtracting an integer multiple of 180° to each value to make both lie between 0 and 360. The *DRM* is that direction, of these two, which is on the side of b_1 to which the bearings draw. Similarly, one can solve (11-2) for b_0.

Proof: Referring to Figure 11.1, apply the law of sines twice to obtain (recall $\sin(180° - x) = \sin x$)

$$\frac{\sin(DRM - b_1)}{h} = \frac{\sin b_{12}}{y},$$

$$\frac{\sin(DRM - b_3)}{h} = \frac{\sin b_{23}}{y t_{23}/t_{12}}.$$

Hence,

$$\frac{\sin(DRM - b_1)}{\sin(b_2 - b_1)} = \frac{t_{23}\sin[(DRM - b_1) - (b_3 - b_1)]}{t_{12}\sin[(b_3 - b_1) - (b_2 - b_1)]}.$$

By applying the formula for the sine of a sum to the numerator and denominator on the right and manipulating the result, one may obtain equation (11-1). Equation (11-2) follows from the perpendicularity of b_0 to *DRM*. The remainder of the theorem follows from the fact that arccot x has two reciprocal values between 0° and 360°.

Application of Theorem 11.3, including the high sensitivity of *DRM* to bearing errors, is illustrated in problems 1, 2, and 3.

Theorem 11.4. One cannot obtain target course, speed, or range from bearings only, no matter how numerous, when own track is linear.

Proof: If the bearings are constant, the target's relative motion is along the bearing line; obviously there is no unique range, speed, or course which produces the observed bearing.

Suppose the bearings are distinct. Let b_0, t_0, and r_0 be bearing, time, and range at closest approach of the two tracks and let w be *relative* speed. Then at any time t_j,

$$\tan b_{0j} = \frac{w}{r_0} t_{0j} \tag{11-4}$$

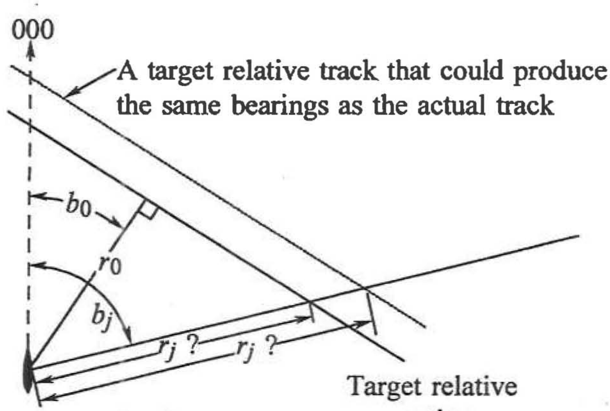
(see Figure 11.2), so

$$b_j = b_0 + \arctan\left(\frac{w}{r_0} t_{0j}\right). \tag{11-5}$$

Equation (11-5) thus gives a complete bearing history and forecast in terms of b_0, t_0, and w/r_0.

If w and r_0 are multiplied by any non-zero constant, (11-4) and (11-5) are unaffected. Thus, there is an infinite family of target tracks each of which produces the same observed bearing history. For each of these tracks, the *DRM* is perpendicular to b_0, the bearing at closest approach. Hence, in relative motion, these tracks are all parallel (see Figure 11.2). At any instant, the ranges of the positions on these tracks are distinct, since the values of r_0, the range at closest approach, are distinct. Therefore, there is no unique range solution. Also, the values of w are distinct, since if two of these tracks had the same relative speed, being separate and parallel, they could not produce the same bearing history. Given that *DRM* is fixed by Theorem 11.3 and own course and speed are known, neither target course nor speed can be unique, because otherwise one could solve a relative motion diagram for a unique value of w. This completes the proof.

FIGURE 11.2. RELATIVE MOTION PLOT



Equation (11-4) can also be used to show that bearings at three distinct times determine all bearings. Given three distinct bearings, equation (11-4) is written for three pairs of values of t_j and b_j, and the equations are solved for t_0, b_0, and w/r_0. The solution is then inserted in (11-5). A better form of this result, however, is given in Theorem 11.7 of 1105.

Some summary remarks are in order on what is learned from bearings when own track as well as target track is linear. Given that constraint, a remarkable result is that the values of own course and speed are not relevant to TMA efforts. After gaining a contact, the first significant TMA information is available after just two bearings (assuming the ideal situation that there is no error in bearings). If the bearings are (approximately) equal, indicating a collision course, this may dictate that own ship commence evasion to avoid collision or coming too close and being counterdetected. For distinct bearings, the important information is whether they draw left or right. This initial bearing drift is the first piece of information from which an approach plan can begin to be formed. The *DRM* can then be determined once three bearings are available. It is important to remember, however, that when both own ship and the target have linear tracks, bearings alone are insufficient to determine target range, course, or speed.

1103 Ekelund Ranging

Once *DRM* is determined and an initial approach plan is developed, the next objective is to get a rough idea of target range. Possibly the best known of all submarine TMA ranging techniques is the Ekelund method of bearings-only ranging. It requires a maneuver by own ship and uses observed bearings before and after a turn by own ship to estimate the target range.

Consider a single leg of own ship's track as in Figure 11.3. The figure shows range and bearing at times t_i and t_j and also distances across the b_i line of sight moved between these times by own ship and target. From the figure one sees that

$$r_j \sin b_{ij} = DTA_{ij} - DOA_{ij}. \tag{11-6}$$

Equation (11-6) will be applied to each leg of a two-leg maneuver. On the first leg, bearings are observed at times t_1 and t_2, and by (11-6)

$$r_2 \sin b_{12} = DTA_{12} - DOA_{12}. \tag{11-7}$$

By dividing both sides by t_{12} and letting t_1 approach t_2, $\sin b_{12}$ approaches b_{12} (in radians) and b_{12}/t_{12} approaches *BR*, defined to be the bearing rate at time t_2. Also $(DTA_{12} - DOA_{12})/t_{12}$ approaches *STA* - *SOA*, where *STA* and *SOA* are defined to be respectively target and own speed across the b_2 line of sight. Thus,

$$r_2 BR = STA - SOA. \tag{11-8}$$

Now suppose, idealistically, that own ship makes an instantaneous turn at time t_2. At this point, *BR* and *SOA* may be interpreted as one-sided derivatives just before t_2. The corresponding one-sided derivatives just after the turn will be designated *BR*′ and *SOA*′. Since it is assumed that the target does not change course or speed, *STA* is the same before and after the turn. Then by the same reasoning that led to (11-8),

$$r_2 BR' = STA - SOA'. \tag{11-9}$$

One eliminates the unknown *STA* from (11-8) and (11-9) to obtain

$$r_2 = \frac{SOA' - SOA}{BR - BR'}. \tag{11-10}$$

Formula (11-10) is the Ekelund range formula: the change at the turn in own speed across line of sight divided by the reverse change in bearing rate. Note that its inputs are entirely measurable on own ship. This is a very convenient estimate of range. It applies at the time of own ship's turn in a two-leg maneuver.

Of course (11-10) must be applied with consistent units, e.g., knots, nm, and radians per hour. To compute *SOA*, it can help to note that

$$SOA = (\text{own speed}) \times \sin(\text{own true course} - \text{true bearing to target}).$$

$$r^* = \frac{1}{.909}\left[8027 + \frac{-.33(88.9-90)+1.14(70.3-85.3)}{-15.0+1.1} - 1.33\right] = 8830 \text{ yards.}$$

The actual range at time 11.3 minutes is 8071 yards. Thus time correction overestimates the range (at time 11.3 minutes) by 9.4 percent, in this example, in comparison with Ekelund ranging which underestimates the range at time 5 minutes, the turn time, by 13.0 percent. Again, exact bearings were used in both methods.

Additional examples of time correction, including effects of bearing errors, are given in problems 11, 12, and 13. In these examples, the improvement of time-corrected Ekelund over Ekelund without time correction is sharper than in the above example.

Note that the development of t^* distinguishes t_1 among the four times t_1, t_2, t_3, and t_4. If instead t_2 were distinguished, for example, then the ensuing r^* would be insensitive to target speed in direction b_2, i.e., to STI_2, rather than to STI_1. With the bearings close enough for the small angle approximations to apply, that change in sensitivity should not be important.

Apart from the choice of the distinguished time, by permutating the indices (1, 2, 3, 4), various values of t^* can be obtained from the four given times and associated bearings. Recall, however, that the above simplified derivation assumes $t_{12} = t_{34}$, so unless also $t_{12} = t_{23}$, there are only four permutations that meet this constraint. Among these, there are only, at most, two different values of t^*. Under derivation without the present simplifications, there are 24 possible permutations, but there are at most twelve different values of t^* for a given set of four observation times. Some of the values of t^* will be too far in the future or past to be tactically useful, but others may be opportune times for own ship to make relatively accurate range estimates which will be insensitive to target speed in the line of sight.

Bearings only TMA is generally more accurate when own ship uses lead-lag, e.g., problems 4 to 14, compared to lag-lead, e.g., the above example.

For most purposes, notably weapon launch, a best range time in the future is highly desirable. (The above time-correction example is an improvement over the Ekelund version of the same example, in this respect as well as in accuracy.) Own ship can effect partial control on this by carefully choosing when and how to maneuver. If, for example, own ship points the target on the first leg and maneuvers to lag the line of sight for the second leg, then a best range time will occur after the last time of the input bearings. Of course, the further into the future the best range time is, the more one must be concerned about a possible course change by the target.

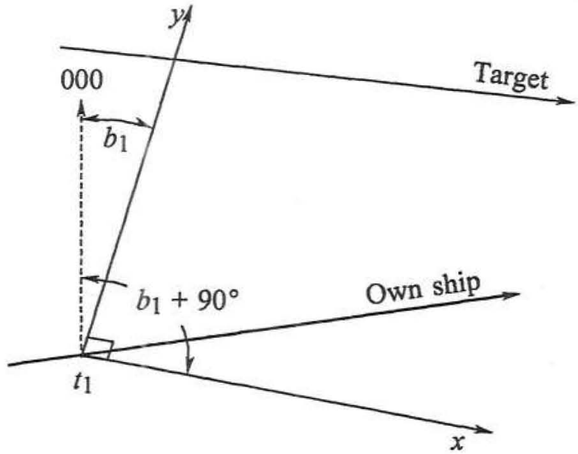
Convenient graphical methods are available for computing t^* under the small angle approximations used here and can be useful in a shipboard tactical situation. Exact formulas without the approximations used here are also available and are easily implemented on a desktop computer.

1105 Spiess TMA

Although not as widely used as Ekelund ranging, another source of range estimation is **Spiess TMA**. Spiess TMA is a technique requiring bearings at four times. Theorem 11.5 below states that given three bearings at distinct times, the locus of target positions at a selected fourth time is a computable straight line. This locus, known as a **Spiess line**, is then simply laid down on a plot and the target position is determined by the intersection of the Spiess line with the observed target bearing at t_4, *unless* the two lines coincide, which is termed a **singularity**. The Spiess line and the fourth bearing line will coincide, if, for example, own track is linear throughout. Once again, target range cannot be determined from bearings only if both own ship and target have linear tracks.

The Spiess range technique will require the following additional notation: As earlier, assume that bearing b_i is observed at time t_i, for $i = 1, 2, \ldots$. Specify a rectangular coordinate system (see Figure 11.4) by fixing the origin at own ship position at time t_1, giving the y-axis direction b_1, and the x-axis direction $b_1 + 90°$. Denote target position at time t_i in these coordinates by (x_i, y_i). The previous double subscript convention is extended to position coordinates: $x_{ij} = x_j - x_i$, etc.

FIGURE 11.4. COORDINATE SYSTEM FOR SPIESS LINE DERIVATION



Theorem 11.5. Given bearings b_1, b_2, b_3 at respective times t_1, t_2, t_3, the locus of target positions at a given fourth time, t_4, with all four times distinct, is a straight line comprised of points (x_4, y_4) such that

$$\begin{aligned} y_4 t_{23} t_{14} &= x_4(t_{13} t_{24} \cot b_{13} - t_{12} t_{34} \cot b_{12}) \\ &+ t_{14} t_{24}(DOI_{13} - DOA_{13} \cot b_{13}) \\ &- t_{14} t_{34}(DOI_{12} - DOA_{12} \cot b_{12}). \end{aligned} \tag{11-28}$$

Proof: Note that

$$y_i - DOI_{1i} = (x_i - DOA_{1i}) \cot b_{1i}, \tag{11-29}$$

for $i = 2$ and 3, and for a constant target speed,

$$\frac{x_{12}}{t_{12}} = \frac{x_{13}}{t_{13}} = \frac{x_{14}}{t_{14}} \quad \text{and} \quad \frac{y_{12}}{t_{12}} = \frac{y_{13}}{t_{13}} = \frac{y_{14}}{t_{14}}, \tag{11-30}$$

where $x_1 = 0$ from the coordinate system. These are six linear equations in $x_2, x_3, x_4, y_1, y_2, y_3, y_4$. Using the four equations in (11-30), x_2, x_3, y_2, and y_3 may be eliminated from the two equations in (11-29). The two resulting equations are then solved simultaneously to eliminate y_1 and the result is (11-28). This completes the proof.

The linear locus of points given by (11-28) is called the **Spiess line** at time t_4. It is denoted L_4, or at an arbitrary time t_j playing the role of t_4, it is denoted L_j. As described above, once L_4 has been determined from three bearings using (11-28), the target position is simply the point of intersection of L_4 with the bearing observed at t_4, *unless* these two lines coincide, which is a singularity. If a singularity does occur, range cannot be obtained from this technique with the available data.

One way to obtain target course and speed is to reverse the order of the four time indices and repeat the above procedure to obtain target position at (the original) t_1. It is then a simple computational or graphical manipulation to obtain course and speed. Another possible technique, although more tedious, is to compute the complete solution to the six equations in (11-29) and (11-30) and also (11-29) for $i = 4$. From these solutions, the target speed and course are given by

$$u = \left[\left(\frac{x_{14}}{t_{14}}\right)^2 + \left(\frac{y_{14}}{t_{14}}\right)^2\right]^{1/2} \quad \text{and} \quad c = b_1 + \operatorname{arccot}\left(\frac{y_{14}}{x_{14}}\right).$$

Spiess TMA is greatly simplified if own track is linear through the three bearings. The next two theorems pertain to this.

Theorem 11.6. If own track is linear, then the Spiess line at a chosen time t_j, corresponding to bearing observations at three other times, is the bearing line observed at t_j.

Proof: If the bearing line and Spiess line at time t_j did not coincide, then it would be possible to find the range at that time by the Spiess TMA procedure, violating Theorem 11.4.

Theorem 11.7. If own track is linear and distinct bearings b_1, b_2, b_3 are observed at respective times t_1, t_2, t_3, then the bearing b_4 at a different arbitrary time t_4, past or future, is found by computing

$$b_4 = b_1 + \operatorname{arccot}\left(\frac{t_{13}t_{24}\cot b_{13} - t_{12}t_{34}\cot b_{12}}{t_{23}t_{14}}\right), \tag{11-31}$$

and adding, if necessary, an integer multiple of 180° to make the sum the value between 000 and 360 which draws in the same direction as b_1 through b_3.

Proof: The Spiess line at t_4 coincides with the bearing line at t_4, by Theorem 11.6. The slope of the Spiess line given by (11-28) is the ratio of the coefficient of y_4 to the coefficient of x_4. This slope may also be determined trigonometrically to be cot b_{14}. Equation (11-31) comes from setting the two expressions for slope equal to each other and solving for b_4. This completes the proof.

Figure 11.5 illustrates Spiess ranging via Theorems 11.6 and 11.7. Own track is linear through three bearings, and a fourth bearing is taken after a port turn. The initial leg is extrapolated to the fourth bearing time, and the bearing at the extrapolated position is calculated by Theorem 11.7. By Theorem 11.6, this extrapolated bearing line is the Spiess line at that time. Theorem 11.6 still applies even though own ship maneuvered, since the maneuver in no way affected the Spiess line, given the bearings and times. Target position at t_4 is the intersection of the actual fourth bearing with the extrapolated fourth bearing (Spiess line).

Figures 11.6 and 11.7 show examples of singularities when own ship's track is not linear. In Figure 11.6, own ship's track is linear through the first three bearings, and a fourth bearing is taken after a maneuver. This time, however, the maneuver is such that the extrapolated bearing and the actual bearing coincide causing no range information to be obtainable. Figure 11.7 is an example of a singularity with two bearings on each of two legs; the Spiess line at any of the four bearing times (with respect to the other three) is necessarily the bearing at that time. In this figure, either of the indicated target tracks would satisfy the bearings, and definitive range information can not be obtained with only the four bearings. Spiess ranging is explored through problems 14 to 21.

FIGURE 11.5. SPIESS RANGE

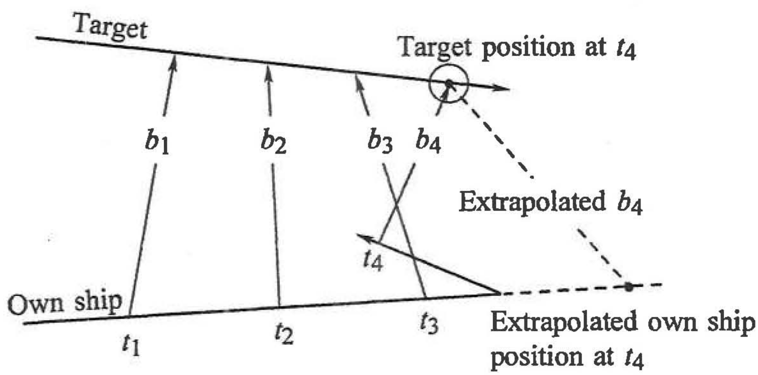
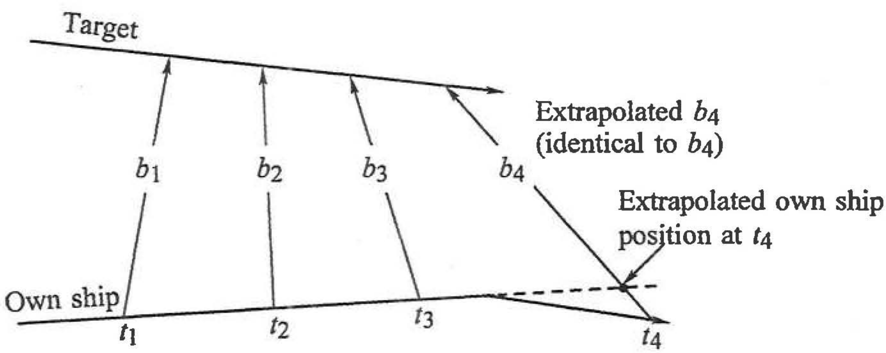


FIGURE 11.6. SINGULARITY IN SPIESS RANGING



The concept of time correction which was discussed in terms of reducing the error of Ekelund range estimations is equally applicable to Spiess ranging. The same formulas used in 1104 apply and the simplifications that $t_{12} = t_{34}$ and the small angle approximations apply are reinstated. As before, the development could be done without these simplifications. It is further assumed that $t_2 = t_3$, so among t_1, t_2, t_3, t_4 there are only three distinct times and they are uniformly spaced.

Referenced to the coordinate system of Figure 11.7, the target position (x_j, y_j) at an arbitrary time t_j is given by

$$x_j = STA_1 \, t_{1j} = DTA_{1j}, \tag{11-32}$$

$$y_j = r_1 + STI_1 \, t_{1j} = r_1 + DTI_{1j}. \tag{11-33}$$

From (11-19) and (11-22),

$$STA_1 = C - D \cdot STI_1, \tag{11-34}$$

where

$$C = \frac{1}{t_{12}}\left(A \cdot b_{12} + DOA_{12} - DOI_{12} b_{12}\right), \tag{11-35}$$

$$D = \frac{b_{12}}{t_{12}}\left(B - t_{12}\right). \tag{11-36}$$

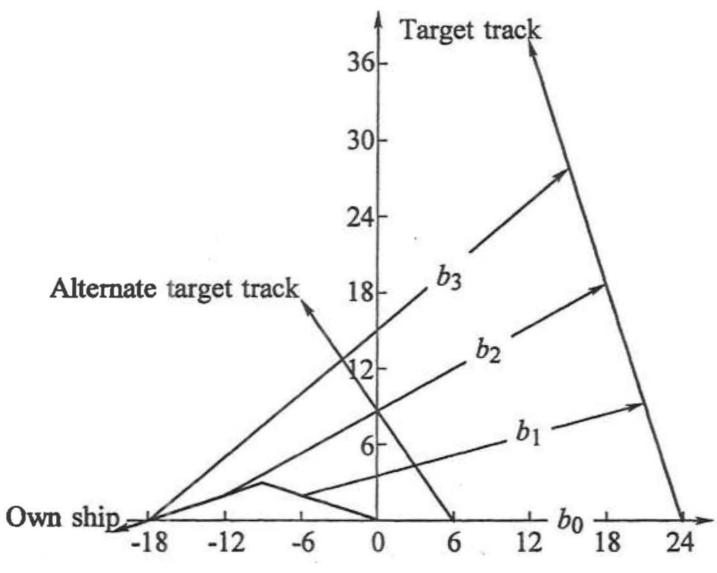
Combining and simplifying (11-32) through (11-36) and (11-22) yields

$$x_j = \left(C - D \cdot STI_1\right)\left(t_j - t_1\right), \tag{11-37}$$

$$y_j = A - \left(B - t_j + t_1\right) STI_1 . \tag{11-38}$$

Note that (11-37) and (11-38) are linear in t_j and are separately linear in STI_1. Now if t_j is fixed as a chosen fourth bearing time, then the set of these (x_j, y_j) points, as STI_1 is varied, is the Spiess line at time t_j. (This proves an alternate version of Theorem 11.5.)

FIGURE 11.7. TWO LEG SPIESS RANGE SINGULARITY



Own ship cannot distinguish between the two constant-velocity targets with bearings only.

Time	0	1	1.5	2	3
Own ship Target track Alt target track	(0, 0) (0, 24) (0, 6)	(2, -6) (11, 21) (5, 3)	(3, -9) — —	(2, -12) (22, 18) (10,0)	(0, 18) (33, 15) (15, -3)

The time quadruple (t_1, t_2, t_3, t_4), with $t_2 = t_3$, and the associated three bearings determine a best range time, t^*. Let L^* be the Spiess line at time t^*. From (11-38) with $t_j = t^* = t_1 + \mathrm{B}$, it follows that $y_j = \mathrm{A}$ *independent of* STI_1. Hence L^* is *perpendicular* to b_1, the y-axis of the coordinate system. If b^* is not far from b_1, this means that L^* and b^* will have a good cross-cut and Spiess ranging at time t^* will be relatively accurate.

Unfortunately, quite the opposite case is also possible. Suppose own track is linear through t_1, $t_2 = t_3$, t_4, and t^*. Then at t^*, the bearing line b^* and Spiess line L^* coincide by Theorem 11.6. Since L^* is perpendicular to b_1, so is b^*. Hence, the $\cos(b^* - b_1)$ factor of the r^* equation, (11-27), equals zero and r^* cannot be computed. This is consistent with Theorem 11.4 and is an inevitable singularity.

An additional observation arises from the linearity of (11-37) and (11-38) in t_j and in STI_1. For a fixed value of STI_1, the (x_j, y_j) values for various values of t_j given by these equations define a linear track. This linear track is the target track which is consistent with the three observed bearings and the chosen value of STI_1. Note the duality: For a fixed t_j, one gets a locus of position as STI_1 is varied; and for a fixed STI_1, one gets a target track as t_j is varied. It can also be shown from (11-37) and (11-38), that at a chosen t_j there is a value of STI_1 whose resultant track coincides with the Spiess line at t_j, and vice versa. More simply, a given bearing triple defines both a set of Spiess lines and a matching set of target tracks. Finally, if the chosen time is the best range time for the triple of bearings, regarded as a quadruple as before, then the corresponding STI_1 is zero.

Figure 11.8 shows the relation between Spiess TMA and time correction when own track is linear throughout. Bearings are taken at times 0, 10, and 30 minutes and a best range time of 48 minutes is calculated from these bearings, regarding the 10-minute observation as both t_2 and t_3. Spiess lines from the three bearings are shown for t_4 = 18, 28, 48(= t^*), and 68. Note how the direction of the Spiess line swings as t_4 changes. At $t_4 = t^* = 48$, L^* is perpendicular to the bearing at time 0, and both own ship and target are on that line. Also shown are target tracks consistent with two separate values of STI_1, +15 knots and -15 knots. If one assumes that the actual STI_1 is between these limits, then the target position at time 48 is limited to the portion of L^* which lies between the 15 knot and -15 knot tracks. Thus, while a complete TMA solution is not possible from bearings only when own track is linear, the target's position can be narrowed down by reasonable assumptions of STI_1.

Figure 11.9 shows a Spiess plot with an own ship turn to port at time 10. Bearings are taken at 0 and 5 minutes on the first leg and at 15 minutes on the second. Spiess lines are shown for various values of t_4, and the target tracks for STI_1 at 15 and -15 knots are also shown. Note how slowly the Spiess lines change direction between times -20 and 5 and between 15 and 40, while direction changes rapidly between 5 and 15. Time 10 is a best range time, and the Spiess line at that time is narrowly delimited by the bounding tracks. Additionally, L^* is roughly perpendicular to the bearing at time 10.

The potential for an accurate estimation of target position at a best range time is clearly demonstrated by this figure. Even without the bearing at time 10, the target's position can be estimated as the intersection of the two possible target tracks since the range at t^* is independent of STI_1. In shipboard use, multiple estimated target solutions are continuously laid out on the plot in an attempt to narrow in on the actual solution. While Figure 11.9 represents a somewhat idealistic plot since there is no bearing scatter and the target conveniently does not maneuver for greater than an hour, all that is required for a rather accurate estimate of target position at $t = 10$ is to lay down the bearing at $t = 10$ minutes and circle the point of intersection with the Spiess line at 10 minutes.

FIGURE 11.8. SPIESS LINES WITH LINEAR OWN SHIP TRACK

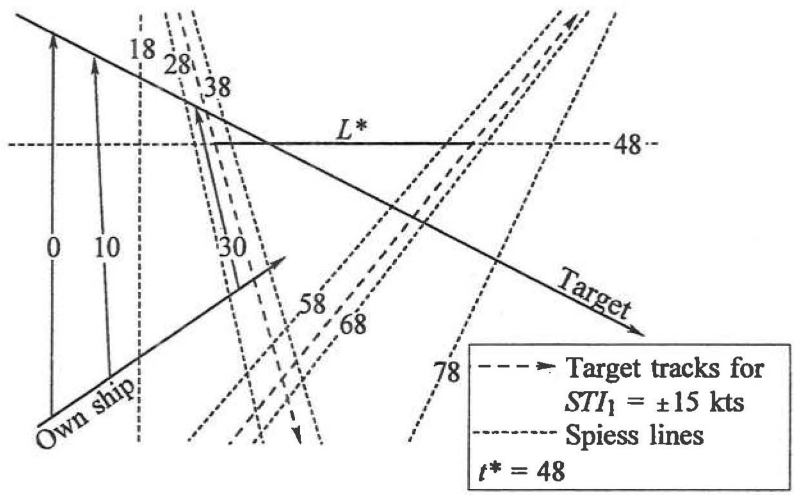
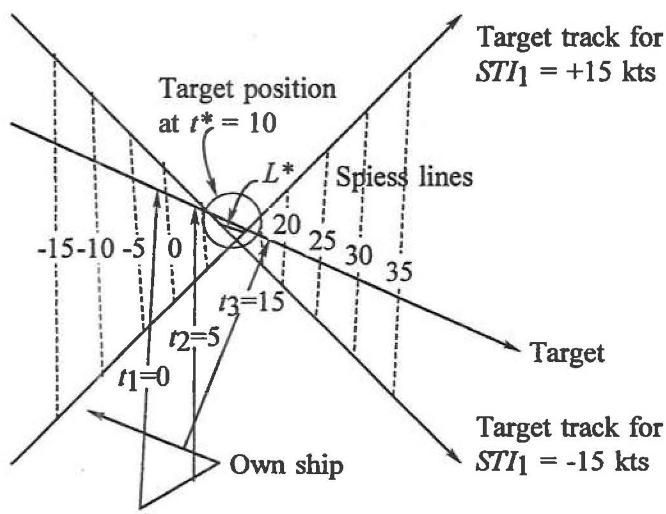


FIGURE 11.9. SPIESS PLOT RELATED TO BEST RANGE TIME



Theorem 11.8, given next without proof, is the "parabola theorem" for bearings-only TMA when own track is linear. From this is derived a generalization to non-linear own track, Corollary 11.9. These results further tie together the above theory.

Theorem 11.8. Suppose own track is linear and distinct bearings at three times are given. A parabola is defined by those bearings and times such that (a) all bearing lines are tangent to the parabola, and (b) so too are the target tracks which are consistent with the bearings for various values of STI_1 as given by (11-37) and (11-38). The axis of this parabola will parallel the *DRM*.

Corollary 11.9. Without assuming that own track is linear, suppose distinct bearings are observed at three times. A parabola is thereby determined which is tangent to all Spiess lines arising from the three bearings and times and to all target tracks consistent with the three bearings and times.

Proof: A linear track is said to **agree** with a bearing if the track position lies on that bearing at the time of the bearing. A point on one of the three bearing lines but not on the others is selected and a linear track through this point which agrees with the three bearings is found. (From linear equations representing the three bearing lines, deduce three equations for the coefficients of the track's equation and solve; these are solvable because the three bearings are distinct.) Theorem 11.7 gives future and past bearing lines from this linear track. Now consider the Spiess line determined by own track, the three bearings and times, and a chosen fourth time. The slope of this Spiess line is the same as that of the bearing line at that time (compare (11-28) and (11-31)). Since the Spiess line and the bearing line both pass through the target and they have the same slope, they coincide. Theorem 11.8 may be applied to find the parabola determined by the three bearings and the linearity of the chosen track. (Although own ship track is not linear, the calculated track is linear so Theorem 11.8 is applicable to it.) As stated in Theorem 11.8, this parabola will be tangent to all the bearing lines. Since the bearings from the linear track and the Spiess lines from own track coincide, this parabola must also be tangent to the Spiess lines. Finally, since the set of tracks consistent with the three bearings coincides with the set of Spiess lines, the parabola must also be tangent to the target tracks. This proves the corollary.

Theorem 11.8 is illustrated by Figure 11.10. Although the parabolic envelope of Spiess lines and target tracks is not explicitly shown in Figures 11.8 and 11.9, a brief reexamination of those figures will show the same principle applies to those figures as well.

As stated early, particular importance is attached to the estimation of range when performing bearings-only TMA. Therefore, the majority of this chapter has been spent in discussing ways of estimating target range. First, it was shown that a complete target solution is not possible if own ship's track is linear. The most common technique of range estimation, Ekelund ranging, and the errors involved in it were discussed next. Also examined was the time correction method to reduce range errors. Finally, the technique of Spiess ranging was explored. Links between time correction and Spiess theory have been found via the parabolic envelope of bearings between linear units. Ekelund ranging, the concept of time correction, and, to a lesser degree, Spiess ranging are not merely techniques used for mathematical discussion. These techniques, and others, have been proven useful in shipboard fire control evolutions, and the creation of these techniques has primarily evolved from shipboard experience.

1106 Other Literature and History

The first complete TMA solution from bearings only was found in 1953 by F. N. Spiess of the Scripps Oceanographic Institution [1], who had considerable submarine war patrol experience in