The Analytic Hierarchy Process and its Generalizations

Thesis by

Edit Adamcsek

Supervisor

Róbert Fullér

Eötvös Loránd University

2008

Contents

Chapter 1

Introduction

The Analytic Hierarchy Process (AHP) is a multi-criteria decision making method developed by Thomas Saaty [8]. AHP allows decision makers to model a complex problem in a hierarchical structure, showing the relationships of the goal, objectives (criteria), and alternatives. AHP is made up of several components such as hierarchical structuring of complexity, pairwise comparisons, judgments, an eigenvector method for deriving weights, and consistency considerations.

In the case of making a decision individually, the best alternative can be easily determined in accordance with the preference of the decision-maker. When the decision is to be decided by a group of people, it is very common that conflicting preferences complicate the evaluation processes leading to an erroneous conclusion. Therefore, it is necessary to aggregate the individual preferences objectively in order to optimize the decision outcomes. To objectify the decision, group decision making is frequently employed.

Many decision problems cannot be structured hierarchically. Not only does the importance of the criteria determine the importance of the alternatives, as in a hierarchy, the importance of the alternatives themselves determine the importance of the criteria. The Analytic Network Process (ANP) provides a solution for problems which can be modelled using a diagram called a network.

Chapter 2

Theoretical foundation - Principles and axioms

AHP is based on three basic principles:

- decomposition,
- comparative judgments, and
- hierarchic composition or synthesis of priorities.

The *decomposition principle* is applied to structure a complex problem into a hierarchy of clusters. The *principle of comparative judgments* is applied to construct pairwise comparisons of all combinations of elements in a cluster with respect to the parent of the cluster. These pairwise comparisons are used to derive local priorities of the elements in a cluster with respect to their parent. The *principle of hierarchic composition or synthesis* is applied to multiply the local priorities of elements in a cluster by the global priority of the parent element, producing global priorities throughout the hierarchy and then adding the global priorities for the lowest level elements (the alternatives).

All theories are based on axioms. The simpler and fewer the axioms, the more general and applicable is the theory. Originally AHP was based on three relatively simple axioms [10]. The first axiom, the *reciprocal axiom*, requires that, if $P_C(A, B)$

is a paired comparison of elements A and B with respect to their parent, element C, representing how many times more the element A possesses a property than does element B, then $P_C(B, A) = \dfrac{1}{P_C(A, B)}$.

The second, or *homogeneity axiom*, states that the elements being compared should not differ by too much, else there will tend to be larger errors in judgment. When constructing a hierarchy of objectives, one should attempt to arrange elements in a cluster so that they do not differ by more than an order of magnitude.

The third, *synthesis axiom* states that judgments about the priorities of the elements in a hierarchy do not depend on lower level elements. This axiom is required for the principle of hierarchic composition to apply and apparently means that the importance of higher level objectives should not depend on the priorities or weights of any lower level factors.

A fourth *expectation axiom*, introduced later by Saaty, says that individuals who have reasons for their beliefs should make sure that their ideas are adequately represented for the outcome to match these expectations. This axiom means that output priorities should not be radically different to any prior knowledge or expectation that a decision maker has.

Chapter 3

The Analytical Process

3.1 Hierarchical Decomposition of the Decision

The first step in using AHP is to develop a hierarchy by breaking the problem down into its components. The three major levels of the hierarchy are the goal, objectives, and alternatives.

Goal The goal is a statement of the overall priority.

Objectives These are the factors needing consideration.

Alternatives We consider the alternatives that are available to reach the goal.

Figure 3.1 shows such a hierarchical structure of factors. The hierarchical structure of the basic AHP allows dependencies among elements to be only between the levels of the hierarchy, and the only possible direction of impact is toward the top of the hierarchy. The elements of a given level are assumed to be mutually independent.

AHP is illustrated with a simple problem. A firm wishes to buy one new piece of equipment of a certain type and has four aspects in mind which will govern its purchasing choice: expense, operability, reliability, and adaptability for other uses, or flexibility. Competing manufacturers of that equipment have offered three options, X, Y and Z. In this example:

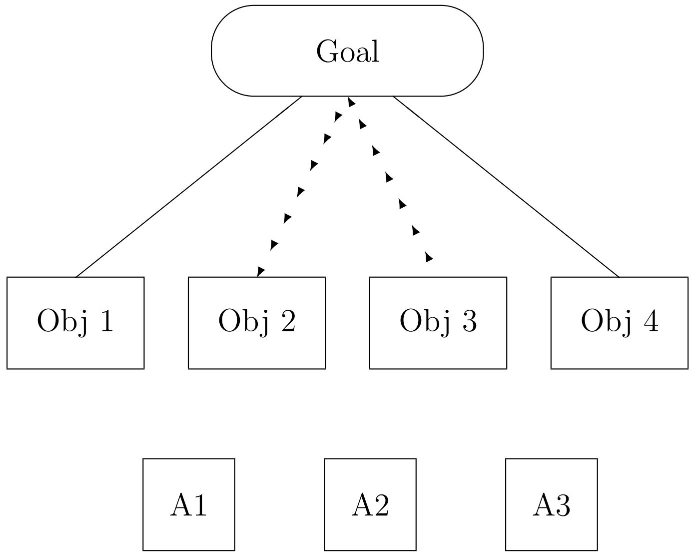

Figure 3.1: An Example of a Hierarchy

- Obj 1 = expense
- Obj 2 = operability
- Obj 3 = reliability
- Obj 4 = flexibility
- A1 = buying the equipment X
- A2 = buying the equipment Y
- A3 = buying the equipment Z

3.2 Pairwise Comparisons

After arranging the problem in a hierarchical fashion, the next step is to establish priorities. Two types of pairwise comparisons are made in the AHP. The first is between pairs of objectives and is used to show our priorities. The second type of pairwise comparison is between pairs of alternatives and is used to determine their relative merits.

3.2.1 Importance of Objectives

Pairwise comparisons of the factors are made in terms of importance. When comparing a pair of objectives, a ratio of relative importance of the factors can be established. This ratio need not be based on some standard scale such as feet or meters but merely represents the relationship of the two factors being compared. In AHP we use the verbal scale to enter judgements. This is essentially an ordinal scale. When a decision maker judges A to be strongly more important than B we know that A is more important than B, but we do not know the interval between A and B or the ratio of A to B.

Intensity of importance	Definition	Explanation
1	Equal importance	Two factors contribute equally to the objective.
3	Somewhat more important	Experience and judgement slightly favour one overthe other.
5	Much more important	Experience and judgement strongly favour one overthe other.
7	Very much more important	Experience and judgement very strongly favour one over the other. Its importance is demonstrated in practice.
9	Absolutely more important	The evidence favouring one over the other is of the highest possible validity.
2,4,6,8	Intermediate values	When compromise is needed.

According to the reciprocal axiom, if attribute A is absolutely more important than attribute B and is rated at 9, then B must be absolutely less important than A and is valued at $\frac{1}{9}$.

What is the relative importance to the management of the firm in our example of the cost of equipment as opposed to its ease of operation? They are asked to choose whether cost is very much more important, rather more important, as important, and so on down to very much less important, than operability. These pairwise comparisons are carried out for all factors to be considered, and the matrix of judgements is completed. We first provide an initial matrix for the firms pairwise comparisons in which the principal diagonal contains entries of 1, as each factor is as important as itself.

	Expence	Operability	Reliability	Flexibility
Expence	1			
Operability		1		
Reliability			1	
Flexibility				1

Let us suppose that the firm decides that operability is slightly more important than cost. In the matrix that is rated as 3 in the cell Operability, Expence and $\frac{1}{3}$ in Expence, Operability. They also decide that cost is far more important than reliability, giving 5 in Expence, Reliability and $\frac{1}{5}$ in Reliability, Expence. The firm similarly judges that operability is much more important than flexibility (rating = 5), and the same judgement is made as to the relative importance of flexibility to reliability. This forms the completed matrix.

	Expence	Operability	Reliability	Flexibility
Expence	1	$\frac{1}{3}$	5	1
Operability	3	1	5	1
Reliability	$\frac{1}{5}$	$\frac{1}{5}$	1	$\frac{1}{5}$
Flexibility	1	1	5	1

3.2.2 Preference of Alternatives with respect to Objectives

We usually evaluate the preference for the alternatives with respect to the objectives before evaluating the importance of the objectives. This approach is recommended so that we get a better understanding of the alternatives just in case our judgments about the importance of the objectives are dependent on the alternatives.

In our example the firms engineers have looked at the options and decided that

- X is cheap and easy to operate but is not very reliable and could not easily be adapted to other uses.
- Y is somewhat more expensive, is reasonably easy to operate, is very reliable but not very adaptable.
- Z is very expensive, not easy to operate, is a little less reliable than Y but is claimed by the manufacturer to have a wide range of alternative uses.

So we now turn to the three potential machines, X, Y and Z. We now need four sets of pairwise comparisons but this time in terms of how well X, Y and Z perform in terms of the four criteria.

Expence	X	Y	Z
X	1	5	9
Y	$\frac{1}{5}$	1	3
Z	$\frac{1}{9}$	$\frac{1}{3}$	1

Operability	X	Y	Z
X	1	1	5
Y	1	1	3
Z	$\frac{1}{5}$	$\frac{1}{3}$	1

Reliability	X	Y	Z
X	1	$\frac{1}{3}$	$\frac{1}{9}$
Y	3	1	$\frac{1}{3}$
Z	9	3	1

Flexibility	X	Y	Z
X	1	$\frac{1}{9}$	$\frac{1}{5}$
Y	9	1	2
Z	5	$\frac{1}{2}$	1

In general we have square and reciprocal matrixes. These are the pairwise comparison matrixes.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},$$

where $a_{ij} = \dfrac{1}{a_{ji}} \quad \forall i, j.$

3.3 Pairwise Matrix Evaluation

Suppose we already know the relative weights of criteria: $w_1, w_2, \dots, w_n$. We can assume that $\sum_{i=1}^{n} w_i = 1$. We can express them in a pairwise comparison matrix as follows:

$$W = \begin{pmatrix} w_{11} & w_{12} & w_{13} & \dots & w_{1n} \\ w_{21} & w_{22} & w_{23} & \dots & w_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \\ w_{n1} & w_{n2} & w_{n3} & \dots & w_{nn} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & \frac{w_1}{w_2} & \frac{w_1}{w_3} & \dots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & 1 & \frac{w_2}{w_3} & \dots & \frac{w_2}{w_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \frac{w_n}{w_3} & \dots & 1 \end{pmatrix}$$

Note that $\forall i, j, k$:

$$w_{ij} = \frac{1}{w_{ji}}$$

and

$$w_{ij} = \frac{w_i}{w_j} = \frac{w_i}{w_k}\frac{w_k}{w_j} = w_{ik}w_{kj}.$$

Such a matrix is called a consistent matrix.

3.3.1 Eigenvector Method

The method below was suggested by Saaty [8]. If we wanted to recover or find the vector of weights, $(w_1, w_2, w_3, ..., w_n)$ given these ratios, we can take the matrix product of the matrix W with the vector $\underline{w}$ to obtain:

$$\begin{pmatrix} 1 & \frac{w_1}{w_2} & \frac{w_1}{w_3} & \dots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & 1 & \frac{w_2}{w_3} & \dots & \frac{w_2}{w_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \frac{w_n}{w_3} & \dots & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} nw_1 \\ nw_2 \\ \vdots \\ nw_n \end{pmatrix}$$

$$W\underline{w} = n\underline{w}$$

If we knew the consisten W matrix, but not $\underline{w}$, we could solve the above for $\underline{w}$. This is an eigenvalue problem: $W\underline{w} = \lambda\underline{w}$. Each row in matrix W is a constant multiple of the first row. For such a matrix, the rank of the matrix is one, and all the eigenvalues of W are zero, except one. Since the sum of the eigenvalues of a positive matrix is equal to the trace of the matrix (the sum of the diagonal elements), the non zero eigenvalue has a value of n, the size of the matrix. Since $W\underline{w} = n\underline{w}$, $\underline{w}$ is the eigenvector of W corresponding to the maximum eigenvalue n.

For matrices involving human judgement, the condition $w_{ij} = w_{ik}w_{kj}$ does not hold as human judgements are inconsistent to a greater or lesser degree. Now we estimate w_{ij} by a_{ij}. $A = [a_{ij}]$ would be the matrix of the pairwise comparisons. In the matrix A the strong consistency property most likely does not hold. Small perturbations in the entries imply similar perturbations in the eigenvalues, thus the eigenvalue problem for the inconsistent case is:

$$A\underline{w} = \lambda_{max}\underline{w},$$

where λ_{max} will be close to n (actually greater than or equal to n) and the other eigenvalues will be close to zero. The estimates of the weights for the activities can be found by normalizing the eigenvector corresponding to the largest eigenvalue in the above matrix equation.

There are other methods than the eigenvector method for estimating weights (w_i) from the matrix of pairwise comparisons ($A = [a_{ij}]$). Here are the presentations of the least squares and logaritmic least squares methods.

3.3.2 Least Squares Method

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} - \frac{w_i}{w_j})^2$$

$$\sum_{i=1}^{n} w_i = 1$$

$$w_i > 0 \quad \text{for } i = 1, \ldots, n$$

3.3.3 Logaritmic Least Squares Method

$$\min \sum_{i<j} \sum_{j=1}^{n} [\ln a_{ij} - \ln(\frac{w_i}{w_j})]^2$$

$$\prod_{i=1}^{n} w_i = 1$$

$$w_i > 0 \quad \text{for } i = 1, \ldots, n$$

Putting $x_i = \ln(w_i)$ and $y_{ij} = \ln(a_{ij})$ we have

$$\sum_{i<j} \sum_{j=1}^{n} [y_{ij} - x_j + x_i]^2$$

Minimizing this we obtain [2]:

$$nx_i - \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_{ij}, \quad i = 1, \ldots, n,$$

which under the side condition

$$\sum_{j=1}^{n} x_j = 0$$

gives

$$x_i = \frac{1}{n} \sum_{j=1}^{n} y_{ij}, \quad i = 1, \ldots, n.$$

So the weights can be expresses in the form

$$w_i = (\prod_{j=1}^{n} a_{ij})^{\frac{1}{n}}.$$

The normalized geometric means of the rows are very close to the eigenvector corresponding to the largest eigenvalue of the matrix. In our example we would like to compute the eigenvector of the following matrix (see on page 11):

$$\begin{pmatrix} 1 & \frac{1}{3} & 5 & 1 \\ 3 & 1 & 5 & 1 \\ \frac{1}{5} & \frac{1}{5} & 1 & \frac{1}{5} \\ 1 & 1 & 5 & 1 \end{pmatrix}$$

The geometric means are computed as:

$$\begin{aligned} m_1 &= \sqrt[4]{1 \times \frac{1}{3} \times 5 \times 1} = \sqrt[4]{\frac{5}{3}} = 1.136 \\ m_2 &= \sqrt[4]{3 \times 1 \times 5 \times 1} = \sqrt[4]{15} = 1.968 \\ m_3 &= \sqrt[4]{\frac{1}{5} \times \frac{1}{5} \times 1 \times \frac{1}{5}} = \sqrt[4]{\frac{1}{125}} = 0.299 \\ m_4 &= \sqrt[4]{1 \times 1 \times 5 \times 1} = \sqrt[4]{5} = 1.495 \end{aligned}$$

With the help of the sum of theese values ($m_1 + m_2 + m_3 + m_4 = 4,898$) we compute the normalized geometric mean, the estimate of the eigenvector:

$$\underline{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} 0.232 \\ 0.402 \\ 0.061 \\ 0.305 \end{pmatrix}$$

These four numbers correspond to the relative values of Expence, Operability, Reliability and Flexibility. The 0.402 means that the firm values operability most of all; 0,305 shows that they like the idea of flexibility; the remaining two numbers show that they not desperately worried about cost and are not interested in reliability. We now turn to the other four matrices (on page 12). We compute the eigenvectors with the same method. The results are shown below:

	Expence	Operability	Reliability	Flexibility
X	0.751	0.480	0.077	0.066
Y	0.178	0.406	0.231	0.615
Z	0.071	0.114	0.692	0.319

This matrix summarises the Performance of the three machines in terms of what the firm wants. Reading down each column, it somewhat states the obvious: X is far better than Y and Z in terms of cost; it is a little better than Y in terms of operability, however, X is of limited value in terms of reliability and flexibility. These are not, however, absolutes; they relate only to the set of criteria chosen by this hypothetical firm.

3.4 Additive Weighted Aggregation of Priorities

Suppose that we have all the weights of criteria and all the performances respect to each criterion. Let $v_1, v_2, \ldots, v_l$ denote the weights of criteria and p_{ij} ($i = 1, \ldots, k$, $j = 1, \ldots, l$) the performance of ith alternative on jth criterion. Now the global priority of the ith alternative can be obtained as the weighted sum of performances:

$$w_i = \sum_{j=1}^{l} v_j p_{ij}$$

In our example we determined the relative weight of factors, and we also have the relative priorities of the alternatives with respect to objectives (see on page 17). We now need to choose between the alternatives, we combine the performance of the machines with the preference of the firm:

	Expence 0.232	Operability 0.402	Reliability 0.061	Flexibility 0.305	global priorities
X	0.751	0.480	0.077	0.066	0.392
Y	0.178	0.406	0.231	0.615	0.406
Z	0.071	0.114	0.692	0.319	0.204

This mean that X, which scores 0.392, seems to come out slightly worse in terms of its ability to meet the firms needs than does Y at 0.406. Z is well behind at 0.204 and would do rather badly at satisfying the firm's requirements in this illustrative case.

3.5 Evaluation of Rating Inconsistency

The final stage is to calculate a Consistency Ratio (CR) to measure how consistent the judgements have been relative to large samples of purely random judgements. If the CR is much in excess of 0.1 the judgements are untrustworthy because they are too close for comfort to randomness and the exercise is valueless or must be repeated.

The closer λ_{max} is to n, the more consistent the judgments. Thus, the difference, $\lambda_{max} - n$, can be used as a measure of inconsistency (this difference will be zero for perfect consistency). Instead of using this difference directly, Saaty defined a Consistency Index (CI) as:

$$\frac{\lambda_{max} - n}{n - 1}$$

since it represents the average of the remaining eigenvalues. In order to derive a meaningful interpretation of either the difference or the consistency index, Saaty simulated random pairwise comparisons for different size matrices, calculating the consistency indices, and arriving at an average consistency index for random judgments for each size matrix. In the table below the upper row is the order of the random matrix, and the lower is the corresponding index of consistency for random judgements.

1	2	3	4	5	6	7
0.00	0.00	0.58	0.9	1.12	1.24	1.32
8	9	10	11	12	13	14
1.41	1.45	1.49	1.51	1.48	1.56	1.57

He then defined the consistency ratio as the ratio of the consistency index for a particular set of judgments, to the average consistency index for random comparisons for a matrix of the same size.

$$CR = \frac{CI}{\text{mean random } CI}$$

Since a set of perfectly consistent judgments produces a consistency index of 0, the consistency ratio will also be zero. A consistency ratio of 1 indicates consistency akin to that, which would be achieved if judgments were not made intelligently, but rather at random. This ratio is called the inconsistency ratio since the larger the value, the more inconsistent the judgments.

In our example the consistency ratio of the matrix, which shows the importance of objectives, is 0.55, well below the critical limit, so the firm is consistent in its choises. The ratios of the other 4 matrices are: 0.072, 0.026, 0, 0. This means, that the firm would buy the equipment Y.

Chapter 4

Group Decision Making

4.1 Aggregating Individual Judgements and Priorities

The Analytic Hierarchy Process is often used in group settings where group members either engage in discussion to achieve a consensus or express their own preferences. When synthesizing the judgments given by n judges we have to condider the followings:

Pareto principle (Unanimity condition) The Pareto principle essentially says that given two alternatives A and B, if each member of a group of individuals prefers A to B, then the group must prefer A to B.

$$f(x, x, \ldots, x) = x,$$

where $f(x_1, x_2, \ldots, x_n)$ is the function for synthesizing the judgments.

Homogeneity condition If all individuals judge a ratio t times as large as another ratio, then the synthesized judgment should also be t times as large.

$$f(tx_1, tx_2, \ldots, tx_n) = tf(x_1, x_2, \ldots, x_n),$$

where $t > 0$.

Reciprocity requirement The synthesized value of the reciprocal of the individual judgments should be the reciprocal of the synthesized value of the original judgments.

$$f(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}) = \frac{1}{f(x_1, x_2, \ldots, x_n)}$$

Theorem 1. *The general synthesizing functions satisfying the unanimity and homogeneity conditions are*
the geometric mean: $f(x_1, x_2, \ldots, x_n) = \sqrt{x_1 x_2 \ldots x_n}$ *and*
the arithmetic mean: $f(x_1, x_2, \ldots, x_n) = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$.
If moreover the reciprocal property is assumed even for a single n-tuple $(x_1, x_2, \ldots, x_n)$ *of judgements of n individuals, where not all* x_k *are equal, then only the geometric mean satisfies all the above conditions.*

Individual judgments can be aggregated in different ways. Two methods are the aggregation of individual judgments and the aggregation of individual priorities. The choice of method depends on whether the group is assumed to act together as a unit or as separate individuals.

4.1.1 Aggregation of Individual Judgements

When individuals are willing to, or must relinquish their own preferences (values, objectives) for the good of the organization, they act in concert and pool their judgments in such a way that the group becomes a new individual and behaves like one. Individual identities are lost with every stage of aggregation and a synthesis of the hierarchy produces the group's priorities. Because we are not concerned with individual priorities, and because each individual may not even make judgments for every cluster of the hierarchy, there is no synthesis for each individual, individual priorities are irrelevant or non-existent. Thus, the Pareto principle is irrelevant. Furthermore, since the group becomes a new individual and behaves like one, the reciprocity requirement for the judgments must be satisfied and the geometric mean rather than an arithmetic mean must be used.

4.1.2 Aggregation of Individual Priorities

When individuals are each acting in his or her own right, with different value systems, we are concerned about each individual's resulting alternative priorities. An aggregation of each individual's resulting priorities can be computed using either a geometric or arithmetic mean. Neither method will violate the Pareto principle: If $a_i \geq b_i$, $i = 1, 2, \ldots, n$ then

$$\sum_{i=1}^{n} \frac{a_i}{n} \geq \sum_{i=1}^{n} \frac{b_i}{n}$$

for an arithmetic mean, and

$$\sqrt[n]{\prod_{i=1}^{n} a_i} \geq \sqrt[n]{\prod_{i=1}^{n} b_i}$$

for a geometric mean provided $a_i \geq 0$ and $b_i \geq 0$, $i = 1, 2, \ldots, n$. While either an arithmetic or geometric mean can be used for aggregating individual priorities, the geometric mean is more consistent with the meaning of priorities in AHP. In particular, preferences in AHP represent ratios of how many times more important (preferable) one factor is than another. Synthesized alternative priorities in AHP are ratio scale measures and have meaning such that the ratio of two alternatives' priorities represents how many times more preferable one alternative is than the other.

4.1.3 Weighted arithmetic and geometric means

When calculating the geometric average of the judgments or either the arithmetic or geometric average of priorities we often assume that the individuals are of equal importance. If, however, group members are not equally important, we can form a weighted geometric mean or weighted arithmetic mean.

Theorem 2. *The general weighted synthesizing functions satisfying the unanimity and homogeneity conditions are*
the weighted geometric mean: $f(x_1, x_2, \ldots, x_n) = x_1^{w_1} x_2^{w_2} \ldots x_n^{w_n}$ *and*
the weighted arithmetic mean: $f(x_1, x_2, \ldots, x_n) = w_1 x_1 + w_2 x_2 + \cdots + w_n x_n$*, where*

$w_1 + w_2 + \cdots + w_n = 1$, $w_i > 0$ ($i = 1, 2, \ldots, n$).
If f also has the reciprocal property and for a single set of entries $(x_1, x_2, \ldots, x_n)$ of judgements of n individuals, where not all x_k are equal, then only the weighted geometric mean applies.

Weighted geometric mean of judgments:

$$J_g(k,l) = \prod_{i=1}^{n} J_i(k,l)^{w_i},$$

where: $J_g(k,l)$ refers to the group judgement of the relative importance of factors k and l, $J_i(k,l)$ refers to individual i's judgment of the relative importance of factors k and l, w_i is the weight of individual i; $\sum_{i=1}^{n} w_i = 1$; and n the number of decision-makers.

Weighted Geometric Mean Complex Judgement Matrix

The weighted geometric mean method is the most common group preference aggregation method in AHP literature.

Definition 1. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times n$ judgement matrices, the* Hardmard product *of A and B can be denoted by $C = A \circ B = c_{ij}$, where $c_{ij} = a_{ij}b_{ij}$, for each i, j.*

Definition 2. *Let $A = (a_{ij})$ be an $n \times n$ judgement matrix, we denote by $A^{\lambda} = (a_{ij}^{\lambda})$ where $\lambda \in \mathbb{R}$.*

Definition 3. *Let $A_1, A_2, \ldots, A_n$ be judgement matrices for the same decision problem, then the weighted geometric mean complex judgement matrix is $\overline{A}$, where*

$$\overline{A} = A_1^{\lambda_1} \circ A_2^{\lambda_2} \circ \cdots \circ A_n^{\lambda_n},$$

$$\sum_{i=1}^{n} \lambda_i = 1 \quad \lambda_k > 0 \quad (k = 1, 2, \ldots, n).$$

Weighted (Un-normalized) geometric mean of priorities:

$$P_g(A_j) = \prod_{i=1}^{n} P_i(A_j)^{w_i},$$

where $P_g(A_j)$ refers to the group priority of alternative j, $P_i(A_j)$ to individual i's priority of alternative j, w_i is the weight of individual i, $\sum_{i=1}^{n} w_i = 1$, and n the number of decision-makers.

Weighted arithmetic mean of priorities:

$$P_g(A_j) = \sum_{i=1}^{n} w_i P_i(A_j),$$

where $P_g(A_j)$ refers to the group priority of alternative j, $P_i(A_j)$ to individual i's priority of alternative j, w_i is the weight of individual i, $\sum_{i=1}^{n} w_i = 1$, and n the number of decision-makers.

4.2 Computing Weights

The question arises as to how to compute the w_i's. Saaty [9] suggests forming a hierarchy of factors such as expertise, experience, previous performance, persuasive abilities, effort on the problem, etc. to determine the priorities of the decision-makers. But who is to provide judgments for this hierarchy? If it cannot be agreed that one person (a supra decision-maker) will provide the judgments, it is possible to ask the same decision-makers who provided judgments for the original hierarchy to provide judgments for this hierarchy as well. If so, we have a meta-problem of how to weight their individual judgments or priorities in the aggregation process to determine the weights for the decision-makers to apply to the aggregation of the original hierarchy. One possibility is to assume equal weights. Ramanathan and Ganesh [7] provide another method, which they call the eigenvector method of weight derivation. They reason that, if $\underline{w} = (w_1, w_2, \ldots, w_n)$ is the true (but unknown) weight priority vector for the individual's weights, and if the individual weight priority vectors derived from the judgments from each of the individuals are arranged in a matrix: $M = (\underline{m_1}, \underline{m_2}, \ldots, \underline{m_n})$, then we can aggregate to find the priorities of the n individuals, $\underline{x}$, where $\underline{x} = M\underline{w}$. Then Ramanathan and Ganesh reason that $\underline{x} = \underline{w}$, resulting in the eigenvector equation: $\underline{w} = M\underline{w}$. We observe that this method is attractive but reasonable only if the weights for obtaining priorities of the decision-makers are assumed to be the same as the weights to be used to aggregate the decision-makers' judgments/priorities for obtaining the alternative priorities in the original hierarchy. In general, this need not be the case.

4.3 The Consistency of the Weighted Geometric Mean Complex Judgement Matrix

Theorem 3. *If judgement matrices $A_1, A_2, \ldots, A_n$ given by experts or decision-makers are of perfect consistency, then the weighted geometric mean complex judgement matrix $\overline{A}$ is of perfect consistency.*

Proof.

$$\overline{A} = A_1^{\lambda_1} \circ A_2^{\lambda_2} \circ \cdots \circ A_n^{\lambda_n} \Rightarrow \overline{a_{ij}} = (a_{ij}^{(1)})^{\lambda_1}(a_{ij}^{(2)})^{\lambda_2} \ldots (a_{ij}^{(n)})^{\lambda_n}$$

Since $A_1, A_2, \ldots, A_n$ are of perfect consistency:

$$(a_{ij}^{(s)})^{\lambda_s} = (a_{ik}^{(s)})^{\lambda_s}(a_{kj}^{(s)})^{\lambda_s} \quad \forall i, j \quad s = 1, \ldots, n.$$

$$\begin{aligned}
\overline{a_{ij}} &= (a_{ij}^{(1)})^{\lambda_1}(a_{ij}^{(2)})^{\lambda_2} \ldots (a_{ij}^{(n)})^{\lambda_n} = \\
&= (a_{ik}^{(1)})^{\lambda_1}(a_{kj}^{(1)})^{\lambda_1}(a_{ik}^{(2)})^{\lambda_2}(a_{kj}^{(2)})^{\lambda_2} \ldots (a_{ik}^{(n)})^{\lambda_n}(a_{kj}^{(n)})^{\lambda_n} = \\
&= (a_{ik}^{(1)})^{\lambda_1}(a_{ik}^{(2)})^{\lambda_2} \ldots (a_{ik}^{(n)})^{\lambda_n}(a_{kj}^{(1)})^{\lambda_1}(a_{kj}^{(2)})^{\lambda_2} \ldots (a_{kj}^{(n)})^{\lambda_n} = \\
&= \overline{a_{ik}a_{kj}}
\end{aligned}$$

Thus $\overline{A}$ is of pervect consistency. □

If the judgement matrices are not of perfect consistency, then the above conclusion does not hold. In this section we prove that $\overline{A}$ is of acceptable consistency ($CR \leq 0.1$) under the condition that each $A_k \quad (k = 1, 2, \ldots, n)$ is of acceptable consistency.

We assume that the components of the real vector of weights ($\underline{w} = (w_1, w_2, \ldots, w_n)$) are perturbed to give the elements of judgement matrix A, namely, $a_{ij} = \dfrac{w_i}{w_j}\epsilon_{ij} \quad \forall i, j$, where $\epsilon_{ij} > 0$ and $\epsilon_{ji} = \dfrac{1}{\epsilon_{ij}}$. The consistency index related to the perturbation matrix $E = (\epsilon_{ij})$ is $CI = \dfrac{\lambda_{max} - n}{n - 1}$, where λ_{max} is the largest eigenvalue of A. $A\underline{w} = \lambda_{max}\underline{w}$, i.e.,

$$\lambda_{max} w_i = \sum_{j=1}^{n} a_{ij} w_j \quad i, j = 1, \ldots, n.$$

Since $a_{ii} = 1$ and $a_{ji} = \dfrac{1}{a_{ij}}$, hence

$$n\lambda_{max} - n = \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} a_{ij} \frac{w_j}{w_i} = \sum_{1 \leq i < j \leq n} (a_{ij} \frac{w_j}{w_i} + a_{ji} \frac{w_i}{w_j}).$$

According to $a_{ij} = \dfrac{w_i}{w_j} \epsilon_{ij}$, we have

$$n\lambda_{max} - n = \sum_{1 \leq i < j \leq n} (\epsilon_{ij} + \epsilon_{ji})$$

$$\lambda_{max} - 1 = \frac{1}{n} \sum_{1 \leq i < j \leq n} (\epsilon_{ij} + \epsilon_{ji}),$$

thus

$$\begin{aligned} CI = & \frac{\lambda_{max} - n}{n - 1} = -1 + \frac{\lambda_{max} - 1}{n - 1} = \\ & = -1 + \frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} (\epsilon_{ij} + \epsilon_{ji}) = \frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} (\epsilon_{ij} + \epsilon_{ji} - 2). \end{aligned}$$

The judgement matrix A can be donated by $A = W \circ E$, where W is a perfectly consistent matrix and E is a perturbation matrix.

Definition 4. *A is of acceptable consistency, if*

$$CI = \frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} (\epsilon_{ij} + \epsilon_{ji} - 2) \leq \alpha,$$

where α is the dead line of acceptable consistent judgement.

In general, α is equal to one-tenth of the mean consistency index of randomly generated matrices, which is given in table on page 19. Since $CR = \dfrac{CI}{\text{mean random } CI}$ we say judgement matrix A is of acceptable consistency, if $CR \leq 0.1$. By definitions on page 23 we have the following properties:

Property 1. $A \circ B = B \circ A$

Property 2. $(A \circ B)^\lambda = A^\lambda \circ B^\lambda, \quad \lambda \in \mathbb{R}$

Property 3. $A \circ B \circ C = (A \circ B) \circ C = A \circ (B \circ C)$

Lemma 1. *Let* $x_i > 0$, $\lambda_i > 0 \quad i = 1, \ldots, n$ *and* $\sum_{i=1}^n \lambda_i = 1$, *then*

$$\prod_{i=1}^n x_i^{\lambda_i} \leq \sum_{i=1}^n \lambda_i x_i$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Proof. This inequality follows the strict convexity of the function $\exp(x)$ and by induction on n:

$$\exp\left\{\sum_{i=1}^n \lambda_i \log x_i\right\} \leq \sum_{i=1}^n \lambda_i \exp(\log x_i),$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$. $\square$

Theorem 4. *Let judgement matrices* $A_1, A_2, \ldots, A_n$ *be of acceptable consistency,* $\lambda_k \in (0,1)$, $\sum_{k=1}^n \lambda_k = 1$, *then the the weighted geometric mean complex judgement matrix* $\overline{A} = A_1^{\lambda_1} \circ A_2^{\lambda_2} \circ \cdots \circ A_n^{\lambda_n}$ *is of acceptable consistency.*

Proof. Since a judgement matrix can be regarded as the matrix obtained by perturbing a consistent matrix, we express the matrices $A_k \quad k = 1, \ldots, n$ as $A_1 = W \circ E_1, A_2 = W \circ E_2, \ldots, A_n = W \circ E_n$, where W is a perfectly consistent matrix, $E_k = (\epsilon_{ij}^{(k)})$ is the perturbation matrix corresponding to A_k. According to the above properties, we can obtain

$$\begin{aligned}
\overline{A} &= A_1^{\lambda_1} \circ A_2^{\lambda_2} \circ \cdots \circ A_n^{\lambda_n} = (A \circ E_1)^{\lambda_1} \circ (A \circ E_2)^{\lambda_2} \circ \cdots \circ (A \circ E_n)^{\lambda_n} = \\
&= A^{\lambda_1} \circ E_1^{\lambda_1} \circ A^{\lambda_2} \circ E_2^{\lambda_2} \circ \cdots \circ A^{\lambda_n} \circ E_n^{\lambda_n} = \\
&= A^{\lambda_1} \circ A^{\lambda_2} \circ \cdots \circ A^{\lambda_n} \circ E_1^{\lambda_1} \circ E_2^{\lambda_2} \cdots \circ E_n^{\lambda_n} = \\
&= A \circ (E_1^{\lambda_1} \circ E_2^{\lambda_2} \cdots \circ E_n^{\lambda_n}).
\end{aligned}$$

By definition we have

$$\frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} (\epsilon_{ij}^{(k)} + \epsilon_{ji}^{(k)} - 2) \leq \alpha \quad k = 1, \ldots, n.$$

Multiplying this by $\lambda_k \in (0,1)$, then

$$\frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} (\lambda_k \epsilon_{ij}^{(k)} + \lambda_k \epsilon_{ji}^{(k)} - 2) \leq \alpha \lambda_k \quad k = 1, \ldots, n.$$

Noting that $\sum_{k=1}^{n} \lambda_k = 1$, it follows that

$$\frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} \left(\sum_{i=1}^{n} \lambda_k \epsilon_{ij}^{(k)} + \sum_{i=1}^{n} \lambda_k \epsilon_{ji}^{(k)} - 2 \right) \leq \alpha. \tag{4.1}$$

From Lemma 1. we have

$$\begin{aligned} &\frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} \left[\prod_{k=1}^{n} \left(\epsilon_{ij}^{(k)} \right)^{\lambda_k} + \prod_{k=1}^{n} \left(\epsilon_{ji}^{(k)} \right)^{\lambda_k} - 2 \right] \leq \\ &\leq \frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} \left[\sum_{k=1}^{n} \lambda_k \epsilon_{ij}^{(k)} + \sum_{k=1}^{n} \lambda_k \epsilon_{ji}^{(k)} - 2 \right]. \end{aligned} \tag{4.2}$$

According to Eqs. 4.1 and 4.2, it can be obtained that

$$\frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} \left[\prod_{k=1}^{n} \left(\epsilon_{ij}^{(k)} \right)^{\lambda_k} + \prod_{k=1}^{n} \left(\epsilon_{ji}^{(k)} \right)^{\lambda_k} - 2 \right] \leq \alpha.$$

By definition it means that $\overline{A} = A_1^{\lambda_1} \circ A_2^{\lambda_2} \circ \cdots \circ A_n^{\lambda_n}$ is of acceptable consistency.

□

4.4 The Logarithmic Least Squares and the Generalized Pseudoinverse in Estimating Ratios

Suppose that a matrix $R = (r_{ijk})$ is available where r_{ijk} is an estimate for the relative significance of the ith and jth factors, provided by a kth decision-maker ($k = 1, \ldots, d_{ij} \leq m \quad \forall i, j$). Moreover let us assume that R is a reciprocal matrix, i.e. $r_{jik} > 0$ and $r_{ijk} = \dfrac{1}{r_{jik}}$. Our purpose is to obtain unique positive estimates $w_1, w_2, \ldots, w_n$ using the logarithmic least squares approach

$$\sum_{i<j} \sum_{k=1}^{d_{ij}} \left[\ln r_{ijk} - \ln \left(\frac{w_i}{w_j} \right) \right]^2.$$

In other words, it is necessary to estimate the following judgment matrix:

$$R = \begin{pmatrix} r_{11,1} & r_{12,1} & \dots & r_{1n,1} \\ \vdots & \vdots & \ddots & \vdots \\ r_{1n,d_{1n}} & r_{1n,d_{1n}} & \dots & r_{1n,d_{1n}} \\ r_{21,1} & r_{22,1} & \dots & r_{2n,1} \\ \vdots & \vdots & \ddots & \vdots \\ r_{2n,d_{2n}} & r_{2n,d_{2n}} & \dots & r_{2n,d_{2n}} \\ \\ \vdots & \vdots & \vdots & \vdots \\ \\ r_{n1,1} & r_{n2,1} & \dots & r_{nn,1} \\ \vdots & \vdots & \ddots & \vdots \\ r_{nn,d_{nn}} & r_{nn,d_{nn}} & \dots & r_{nn,d_{nn}} \end{pmatrix}$$

by a matrix of ratios:

$$W = \begin{pmatrix} 1 & \frac{w_1}{w_2} & \frac{w_1}{w_3} & \dots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & 1 & \frac{w_2}{w_3} & \dots & \frac{w_2}{w_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \frac{w_n}{w_3} & \dots & 1 \end{pmatrix}.$$

The matrix R is defined in the informal way from the mathematical point of view, i.e. it is an n-dimensional square matrix with each entry consisting of expert opinions which number depends on the number of opinions concerning a given pair of factors. In particular cases, R may have empty entries, when an expert refuses to provide his opinion concerning a pair or pairs of factors.

Putting $x_i = \ln(w_i)$ and $y_{ijk} = \ln(r_{ijk})$ we consider

$$\sum_{i<j} \sum_{k=1}^{d_{ij}} [y_{ijk} - x_j + x_i]^2.$$

Minimizing this we obtain:

$$x_i \sum_{j \neq ij=1}^{n} d_{ij} - \sum_{j \neq ij=1}^{n} d_{ij} x_j = \sum_{j \neq ij=1}^{n} \sum_{k=1}^{n} y_{ijk}, \quad i = 1, \dots, n, \tag{4.3}$$

where

$$d_{ij} > 0 \quad \forall i, j,$$

$$\sum_{j \neq i j=1}^{n} d_{ij} > 0 \quad \forall i.$$

The set of equations 4.3 may be written in a matrix form:

$$\mathbf{Ax} = \mathbf{b},$$

where A is a real symmetric matrix with rows and columns summing up to zero and b is a real vector with entries summing up to zero as well. The rank of the matrix A is less than n. The matrix A has the following structure:

$$A = \begin{pmatrix} \sum_{j \neq i j=1}^{n} d_{1j} & -d_{12} & \dots & -d_{1n} \\ -d_{21} & \sum_{j \neq i j=1}^{n} d_{2j} & \dots & -d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -d_{n1} & -d_{n2} & \dots & \sum_{j \neq i j=1}^{n} d_{nj} \end{pmatrix}$$

4.4.1 The Logarithmic Least Squares Approach

In case every decision maker has provided his opinion for each pair of factors ($d_{ij} = D \forall i, j$) eqs. 4.3 may be rewritten as

$$x_i \sum_{j \neq i j=1}^{n} D - \sum_{j \neq i j=1}^{n} D x_j =$$

$$n D x_i - D \sum_{j \neq i j=1}^{n} x_j = \sum_{j \neq i j=1}^{n} \sum_{k=1}^{n} y_{ijk} \quad i = 1, \dots, n,$$

which together with the condition $\sum_{j=1}^{n} x_j = 0$ gives

$$x_i = \frac{1}{nD} \sum_{j \neq i j=1}^{n} \sum_{k=1}^{n} y_{ijk}.$$

Thus

$$w_i = \left(\prod_{j=1}^{n} \prod_{k=1}^{D} r_{ijk} \right)^{\frac{1}{nD}} i = 1, \dots, n.$$

Since $\sum_{j=1}^{n} x_j = 0$, $\prod_{i=1}^{n} w_i = 1$. This gives the geometric normalization property for the solution of the problem. When we have the same number of judgments per pair of factors, the geometric mean method may be used and the solution is geometrically normalized.

4.4.2 The Generalized Approach

In order to solve $Ax = b$ we try a more general approach, using the generalized pseudoinverse method. Kwiesielewicz [6] showed that for every real symmetric matrix the spectral decomposition (SD) may be used to define the generalized pseudoinverse matrix (Theorem 7). Moreover when rows and columns of a real symmetric matrix sum up to zero then rows and columns of its pseudoinverse sum up to zero as well (Theorem8).

The Generalized Pseudoinverse Matrix

Since A is a real symmetric matrix, it can be diagonalized by an orthogonal matrix:

$$Q^{-1}AQ = \Lambda.$$

The columns of Q contain complete set of orthonormal eigenvectors and Λ is a diagonal matrix with eigenvalues of A. Q is an orthogonal real matrix, $Q^{-1} = Q^T$, the above equation may be written

$$A = Q\Lambda Q^T.$$

Definition 5. *Let the matrix A^+ be*

$$A^+ = Q\Lambda^+ Q^T,$$

where Λ^+ is a diagonal matrix with

$$\lambda_i^+ = \begin{cases} 1/\lambda_i & if \lambda_i \neq 0, \\ 0 & if \lambda_i = 0. \end{cases}$$

This definition gives a a spectral decomposition (SD) of A^+.

Theorem 5. *For every real m x n matrix A there exists a unique solution X to the following system of matrix equations:*

1. $AXA = A$,
2. $XAX = X$,
3. $(AX)^T = AX$,
4. $(XA)^T = XA$.

Definition 6. *The generalized pseudoinverse of A, denoted as A^+, is the unique matrix X determined by axioms 1.-4.*

Theorem 6. *The matrix A^+ defined by Definition 5 is the generalized pseudoinverse in the sense of Definition 6.*

The following theorem follows from the above:

Theorem 7. *For every real symmetric matrix A there exists a unique pseudoinverse matrix defined by $A^+ = Q\Lambda^+Q^T$.*

Generalized inverse for a matrix with columns and rows summing up to zero

Theorem 8. *The generalized pseudoinverse of every real symmetric matrix with rows summing up to zero, has rows and columns summing up to zero too.*

Problem Solution

Theorem 9. *A necessary and sufficient condition for the equation*

$$Ax = b$$

to have a solution is

$$AA^+b = b,$$

in which case the general solution is

$$x = A^+b + (I - A^+A)y,$$

where y is an arbitrary vector.

The minimum norm solution is

$$x = A^+ b.$$

Since rows and columns of the pseudoinverse sum up to zero, if the minimum norm solution exists, it satisfies the following condition:

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^+ b_j = \sum_{j=1}^{n} b_j \sum_{i=1}^{n} a_{ij}^+ = 0.$$

So after coming back to exponentials, we get $\prod_{i=1}^{n} w_i = 1$. Recall that $w_i = e^{x_i} \quad (i = 1, \ldots, n)$. Now we are ready to state the main conclusion:

Theorem 10. *The minimum norm solution $x = A^+ b$ of the equations set $Ax = b$ if exists gives for the w's the geometric normalization condition $w_i = e^{x_i} \quad i = 1, \ldots, n$.*

So the solution of our problem is geometrically normalized and consistent with the one decision-maker case.

Chapter 5

The Analytic Network Process

5.1 Network

Many decisions cannot be structured hierarchically because they involve the interaction and dependence of higher-level elements on lower-level elements. Not only does the importance of the criteria determine the importance of the alternatives as in a hierarchy, but also the importance of the alternatives themselves determines the importance of the criteria.

Consider the following example. Suppose you are the mayor of a medium size city. The city council has just approved funding for a bridge that will connect the eastern and southern districts saving the residents 30 minutes in commuting time. You announce that the winning proposal will be chosen using a formal evaluation methodology in which the proposals will be evaluated on the basis of strength and aesthetics. In order to be fair, you will, before receiving any bids, specify which of the two objectives will be more important. It seems obvious that strength is much more important than aesthetics and you publicly announce that strength will be the most important objective in choosing the winning proposal. Subsequently, two alternative designs are proposed for the new bridge. Bridge A is extremely save (as safe as any bridge yet built in the State) and beautiful. Bridge B is twice as strong as bridge A, but is ugly. Your hands are tied you have announced that the most important objective is strength and you must choose the ugly bridge. The bridge is

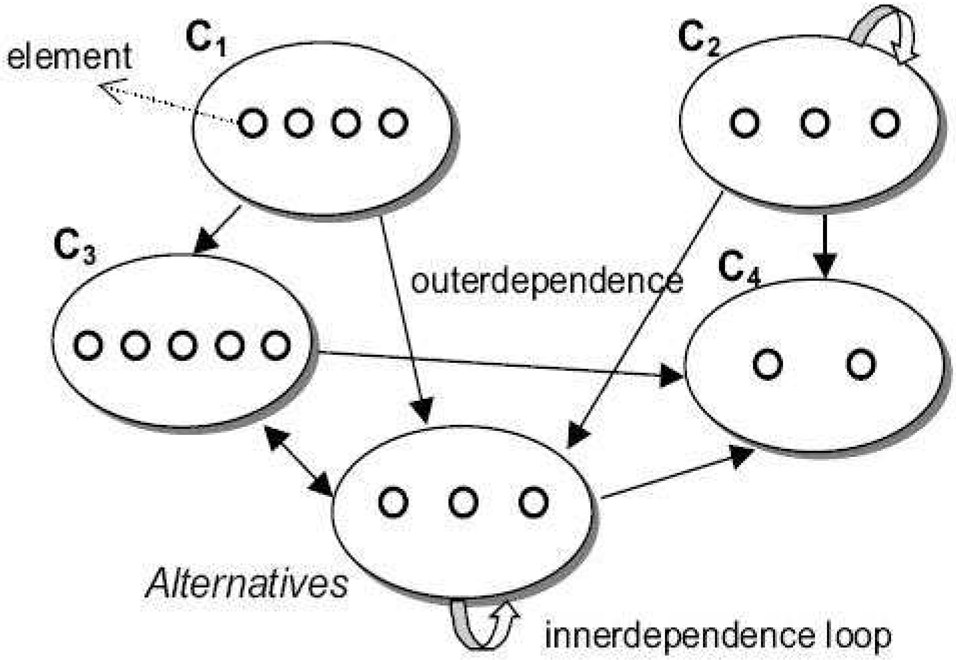

Figure 5.1: Network

built and many town residents are reminded of your decision at least twice a day. You lose the next election and will be wary of formal evaluation methodologies for the rest of your life.

The feedback structure does not have the linear top-to-bottom form of a hierarchy but looks more like a network, with cycles connecting its clusters of elements, which we can no longer call levels, and with loops that connect cluster to itself. A decision problem involving feedback arises often in practice. The Analytic Network Process provides a solution for problems that can be modelled using a diagram called a network, as presented in Figure 5.1.

A network has clusters of elements, with the elements in one cluster being connected to elements in another cluster (outer dependence) or the same cluster (inner dependence). A network is concerned with all the influences that can affect an outcome. It is a model of continual change because everything affects everything else and what we do now can change the importance of the criteria that control the evolution of the outcome.

5.2 Supermatrix

The first phase of the ANP is to compare the criteria in whole system to form the supermatrix. Assume that we have a system of N clusters or components whereby the elements in each component interact or have an impact on or are influenced by some or all of the elements of that component or of another component with respect to a property governing the interactions of the entire system. Assume that the hth component, denoted by C_h $(h = 1, \ldots, N)$, has n_h elements, which we denote by $e_{h1}, e_{h2}, \ldots, e_{hn_h}$. The impact of a given set of elements in a component on another element in the system is represented by a ratio scale priority vector derived from paired comparisons in the usual way. Each priority vector is derived and introduced in the appropriate position as a column vector in a supermatrix of impacts (with respect to one control criterion). W block matrix consists of the collection of the priority weight vectors (w) of the influence of the elements in the ith cluster with respect to the jth cluster. If the ith cluster has no influence to the jth cluster then $W_{ij} = 0$. The matrix obtained in this step is called the supermatrix. The general form of a supermatrix is shown below.

$$
\begin{array}{cc|cccc}
 & & C_1 & C_2 & \cdots & C_N \\
 & & e_{11}e_{12}\ldots e_{1n_1} & e_{21}e_{22}\ldots e_{2n_2} & \cdots & e_{N1}e_{N2}\ldots e_{Nn_N} \\
\hline
 & e_{11} & & & & \\
 & e_{12} & & & & \\
C_1 & \vdots & W_{11} & W_{12} & \cdots & W_{1N} \\
 & e_{1n_1} & & & & \\
 & e_{21} & & & & \\
 & e_{22} & & & & \\
C_2 & \vdots & W_{21} & W_{22} & \cdots & W_{2N} \\
 & e_{2n_2} & & & & \\
 & \vdots & \vdots & \vdots & & \vdots \\
 & e_{N1} & & & & \\
 & e_{N2} & & & & \\
C_N & \vdots & W_{N1} & W_{N2} & \cdots & W_{NN} \\
 & e_{Nn_N} & & & &
\end{array}
$$

The supermatrix, which is composed of ratio scale priority vectors derived from pairwise comparison matrices and the zero vectors, must be stochastic (each column sums to one) to obtain meaningful limiting results. In general the supermatrix is rarely stochastic because, in each column, it consists of several eigenvectors which each sums to one, and hence the entire column of the matrix may sum to an integer greater than one. The natural thing to do, which we all do in practice, is to determine the influence of the clusters on each cluster with respect to the control criterion. This yields an eigenvector of influence of all the clusters on each cluster. The eigenvector obtained from cluster level comparison with respect to the control criterion is applied as the cluster weights. This results in a matrix which each of its columns sums to unity. If any block in the supermatrix contains a column that every element is zero, that column of the supermatrix must be normalized after weighting by the cluster's weights to ensure the column sum to be unity. The concept is similar to Markov Chain that the sum of the probabilities of all states equal to one. This matrix is called the stochastic matrix or weighted supermatrix.

Next, we raise the weighted supermatrix to limiting powers such as $\lim_{k\to\infty} W^k$

to get the global priority vectors or called weights. If the supermatrix has the effect of cyclicity, there may be two or more N limiting supermatrices. In this case, the Cesaro sum is calculated to get the priority. The Cesaro sum is formulated as

$$\lim_{k \to \infty} \left(\frac{1}{N} \right) \sum_{k=1}^{N} W^k$$

to calculate the average effect of the limiting supermatrix (i.e. the average priority weights).

In order to show the concrete procedures of the ANP, a simple example of system development is demonstrated to derive the priority of each criterion. As we know, the key to develop a successful system depending on the match of human and technology factors. Assume the human factor can be measured by the criteria of business culture (C), end-user demand (E) and management (M). On the other hand the technology factor can be measured by the criteria of employee ability (A), process (P) and resource (R). In addition, human and technology factors are affected each other as like as the structure shown in Figure 5.2.

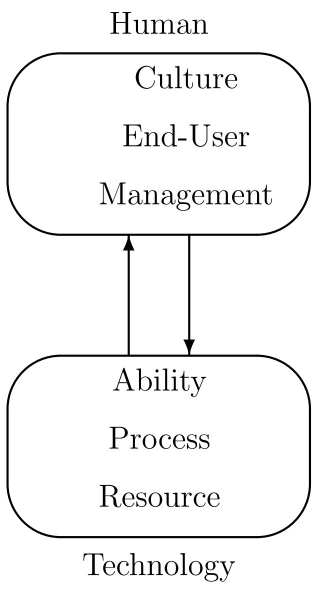

Figure 5.2: Structure

The first step of the ANP is to compare the importance between each criterion.

For example, the first matrix on Table 5.1 is to ask the question "For the criterion of employee ability, how much the importance does one of the human criteria than another". The other matrices can easily be formed with the same procedures. The next step is to calculate the influence (i.e. calculate the principal eigenvector) of the elements (criterion) in each component (matrix).

Now, we can form the supermatrix. Since the human factor can affect the technology factor, and vise versa, the supermatrix is formed as follows:

	C	E	M	A	P	R
C	0	0	0	0.634	0.250	0.400
E	0	0	0	0.192	0.250	0.200
M	0	0	0	0.174	0.500	0.400
A	0.637	0.582	0.136	0	0	0
P	0.105	0.109	0.654	0	0	0
R	0.258	0.309	0.210	0	0	0

Then, the weighted supermatrix is obtained by ensuring all columns sum to unity exactly. It is the same as the supermatrix. Last, by calculating the limiting power of the weighted supermatrix, the limiting supermatrix is obtained as follows:

	C	E	M	A	P	R
C	0	0	0	0.464	0.464	0.464
E	0	0	0	0.210	0.210	0.210
M	0	0	0	0.324	0.324	0.324
A	0.463	0.463	0.463	0	0	0
P	0.284	0.284	0.284	0	0	0
R	0.253	0.253	0.253	0	0	0

when k is even and

	C	E	M	A	P	R
C	0.464	0.464	0.464	0	0	0
E	0.210	0.210	0.210	0	0	0
M	0.324	0.324	0.324	0	0	0
A	0	0	0	0.463	0.463	0.463
P	0	0	0	0.284	0.284	0.284
R	0	0	0	0.253	0.253	0.253

when k is odd.

As we see, the supermatrix has the effect of cyclicity, and the Cesaro sum (i.e. add the two matrices and dividing by two) is used here to obtain the final priorities as follows:

	C	E	M	A	P	R
C	0.233	0.233	0.233	0.233	0.233	0.233
E	0.105	0.105	0.105	0.105	0.105	0.105
M	0.162	0.162	0.162	0.162	0.162	0.162
A	0.231	0.231	0.231	0.231	0.231	0.231
P	0.142	0.142	0.142	0.142	0.142	0.142
R	0.127	0.127	0.127	0.127	0.127	0.127

In this example, the criterion of culture has the highest priority (0.233) in system development and the criterion of end-user has the least priority (0.105).

Ability	Culture	End-user	Management	Eigenvector	Normalization
Culture	1	3	4	0.634	0.634
End-user	1/3	1	1	0.192	0.192
Management	1/4	1	1	0.174	0.174
Process					
Culture	1	1	1/2	0.250	0.250
End-user	1	1	1/2	0.250	0.250
Management	2	2	1	0.500	0.500
Recourse					
Culture	1	2	1	0.400	0.400
End-user	1/2	1	1/2	0.200	0.200
Management	1	2	1	0.400	0.400
Culture	Ability	Process	Resource		
Ability	1	5	3	0.637	0.637
Process	1/5	1	1/3	0.105	0.105
Resource	1/3	3	1	0.258	0.258
End-user					
Ability	1	5	2	0.582	0.582
Process	1/5	1	1/3	0.109	0.109
Resource	1/2	3	1	0.309	0.309
Management					
Ability	1	1/1	1/3	0.136	0.136
Process	5	1	3	0.654	0.654
Resource	3	1/3	1	0.210	0.210

Table 5.1: Pairwise Comparisons

Bibliography

[1] Geoff Coyle: Practical Strategy: Structured tools and techniques [2004] *Financial Times Press*

[2] Crawford, G., and Williams, C, [1985] A note on the analysis of subjective judgment matrices, *Journal of Mathematical Psychology* 29: 387-405.

[3] Ernest H. Forman & Mary Ann Selly [2001] Decision by Objectives *World Scientific*

[4] Ernest Forman, Kirti Peniwati [1998] Aggregating individual judgments and priorities with the Analytic Hierarchy Process *European Journal of Operational Research* 108: 165-169

[5] Jih-Jeng Huang, Gwo-Hshiung Tzeng, Chorng-Shyong Ong, [2004] Multidimensional data in multidimensional scaling using the analytic network process, *Pattern Recognition Letters* 26 (2005) 755-767

[6] Kwiesielewicz, M., [1996] The logarithmic least squares and the generalized pseudoinverse in estimating ratios, *European Journal of Operational Research* 93: 611-619

[7] Ramanathan, R., Ganesh, L.S., [1994] Group Preference Aggregation Methods Employed in AHP: An Evaluation and Intrinsic Process for Deriving Members' Weightages, *European Journal of Operational Research* 79: 249-265.

[8] Saaty, Thomas L. [1980] The Analytic Hierarchy Process, *McGraw-Hill, New York*

[9] Saaty, Thomas L. [1984] Fundamentals of Decision Making and Priority Theory with The Analytic Hierarchy Process, *RWS Publications*

[10] Saaty, Thomas L. [1986] Axiomatic foundations of the AHP, *Management Science* 32: 841-855.

[11] Saaty, Thomas L. [1999] The seven pillars of the Analytic Hierarchy Process

[12] Saaty, T. L. [1999] Fundamentals of the Analytic Network Process, *Proceedings of ISAHP 1999, Kobe*

[13] Z. Xu [2000] On consistency of the weighted geometric mean complex judgement matrix in AHP, *European Journal of Operational Research* 126: 683-687