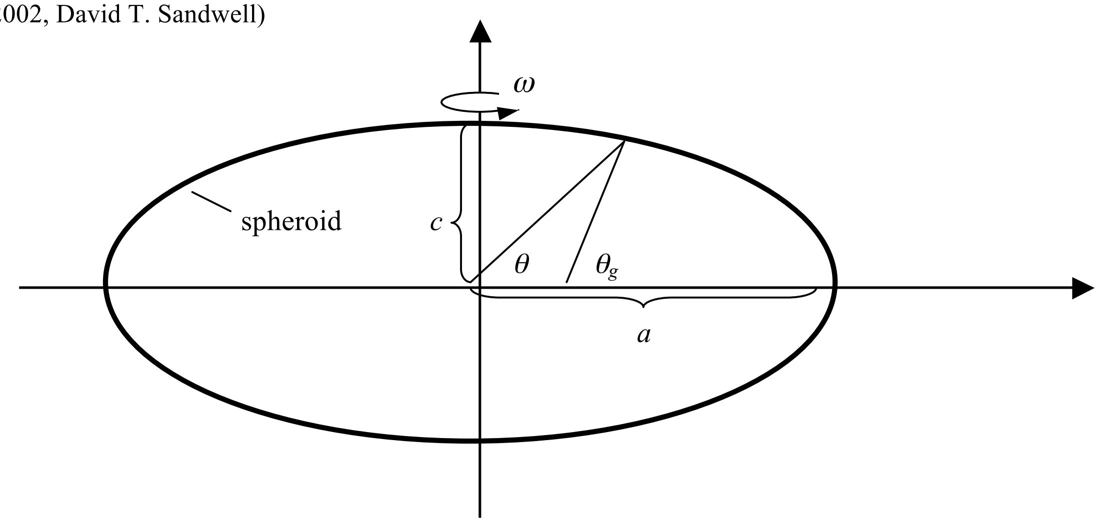
Reference Earth Model - WGS84



parameter	description	formula	value/unit
GM_e	(WGS84)		$3.986004418 \times 10^{14}$ $m^3 s^{-2}$
M_e	mass of earth	-	5.98×10^{24} kg
G	gravitational constant	-	6.67×10^{-11} m^3 kg^{-1} s^{-2}
a	equatorial radius (WGS84)	-	6378137 m
c	polar radius (derived)	-	6356752.3 m
ω	rotation rate (WGS84)	-	7.292115×10^{-5} rad s^{-1}
f	flattening (WGS84)	$f = (a - c)/a$	1/298.257223560
J_2	dynamic form factor (derived)	-	1.081874×10^{-3}
θ_g	geographic latitude	-	-
θ	geocentric latitude	-	-

Radius of spheroid

$$r(\theta) = \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{c^2} \right)^{-1/2} \cong a\left(1 - f\sin^2\theta\right) \qquad (1)$$

Conversion between geocentric θ and geographic θ_g latitude

$$\tan\theta = \frac{c^2}{a^2}\tan\theta_g \quad \text{or} \quad \tan\theta = (1 - f)^2 \tan\theta_g \qquad (2)$$

Gravitational potential in frame rotating with the Earth

$$U_o = -\frac{GM_e}{r} + \frac{GM_e J_2 a^2}{2r^3}\left(3\sin^2\theta - 1\right) - \frac{1}{2}\omega^2 \mathrm{r}^2 \cos^2\theta \qquad (3)$$

Calculation of the second degree harmonic, J_2 from WGS84 parameters

$$J_2 = \frac{2}{3}f - \frac{a^3\omega^2}{3GM_e} \qquad (4)$$

Calculation of J_2 from the polar-C and equatorial-A moments of inertia

$$J_2 = \frac{C - A}{M_e a^2} \qquad (5)$$

Kepler's third law relating orbit frequency-ω_s , and radius-r, to M_e

$$\omega_s^2 r^3 = GM_e \qquad (6)$$

Measurement of J_2 from orbit frequency-ω_s , radius-r, inclination-i, and precession rate-ω_p

$$\frac{\omega_p}{\omega_s} = \frac{-3a^2}{2r^2} J_2 \cos i \qquad (7)$$

Hydrostatic flattening is less than observed flattening

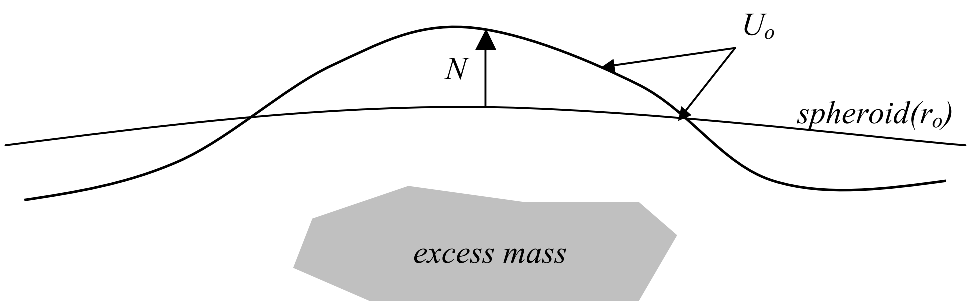
$$f_H = \frac{1}{299.5} < f = \frac{1}{298.257} \qquad (8)$$

Disturbing potential and geoid height

To a first approximation, the reference potential U_o is constant over the surface of the earth. Now we are concerned with deviations from this reference potential. This is called the disturbing potential Φ and over the oceans the anomalous potential results in a deviation in the surface away from the spheroid.

$$U = U_o + \Phi \qquad (9)$$

Where the reference potential U_o is given in equation (3). The *geoid* is the equipotential surface of the earth that coincides with the sea surface when it is undisturbed by winds, tides, or currents. The *geoid height* is the height of the geoid above the spheroid and it is expressed in meters. Consider the following mass anomaly in the earth and its effect on the ocean surface.



Because of the excess mass, the potential on the spheroid is higher than the reference level $U = U_o + \Phi$. Thus, the ocean surface must move further from the center of the earth to remain at the reference level U_o. To determine how far it moves, expand the potential in a Taylor series about the radius of the spheroid at r_o.

$$U_o(r) = U(r_o) + \frac{\partial U}{\partial r}(r - r_o) + \ \ldots \qquad (10)$$

Notice that $g = -\delta U/\delta r$ so we arrive at

$$U(r) - U_o \cong g(r - r_o) \qquad (11)$$

$$\Phi = gN$$

This is Brun's formula that relates the disturbing potential to the geoid height *N*.

Reference gravity and gravity anomaly

The reference gravity is the value of total (scalar) acceleration one would measure on the spheroid assuming no mass anomalies inside of the earth.

$$\mathbf{g} = -\nabla U_o = -\frac{\partial U_o}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial U_o}{\partial \theta}\hat{\theta} - \frac{1}{r\cos\theta}\frac{\partial U_o}{\partial \phi}\hat{\phi} \tag{12}$$

The total acceleration on the spheroid is

$$g = -\left[\left(\frac{\partial U_o}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial U_o}{\partial \theta}\right)^2\right]^{1/2} \tag{13}$$

The second term on the right side of equation (13) is negligible because the normal to the ellipsoid departs from the radial direction by a small amount and the square of this value is usually unimportant. The result is

$$g(r,\theta) = -\frac{GM_e}{r^2}\left[1 - \frac{3J_2a^2}{2r^2}\left(3\sin^2\theta - 1\right)\right] + \omega^2 r\cos^2\theta \tag{14}$$

To calculate the value of gravity anomaly on the spheroid, we substitute

$$r(\theta) = a\left(1 - f\sin^2\theta\right) \tag{15}$$

After substitution, expand the gravity in a binomial series and keep terms of order f but not f^2.

$$g(\theta) = g_e\left[1 + \left(\frac{5}{2}m - f\right)\sin^2\theta\right] \tag{16}$$

$$m = \frac{\omega^2 a^2}{GM_e}$$

The parameter g_e is the value of gravity on the equator and m is approximately equal to the ratio of centrifugal force at the equator to the gravitational acceleration at the equator. In practice, geodesists get together at meetings of the International Union of Geodesy and Geophysics (IUGG) and agree on such things as the parameters of WGS84. In addition, they define something called the *international gravity formula*.

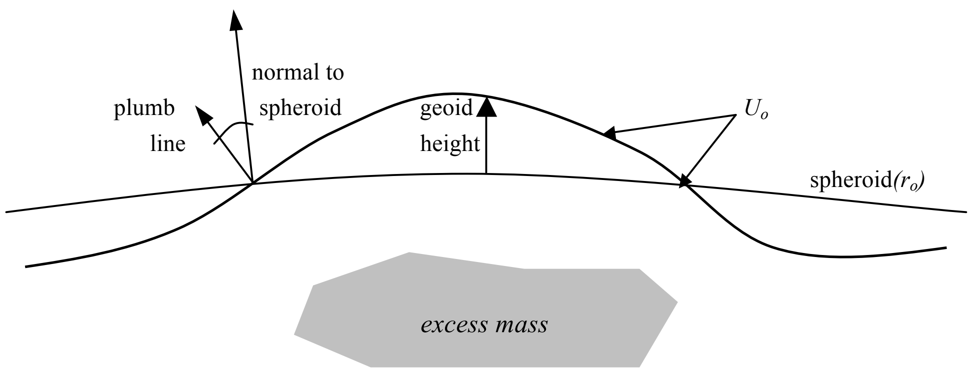
$$g_o(\theta) = 978.03185\left(1 + 0.00527889\sin^2\theta + 0.000023462\sin^4\theta\right) \tag{17}$$

This version was adopted in 1967 so for a real application, you should use a more up-to-date value or use a value that is consistent with all of the other data in your data base.

Free-air gravity anomaly
The free-air gravity anomaly is the negative radial derivative of the disturbing potential but it is also evaluated in the geoid. The formula is

$$\Delta g = -\frac{\partial \Phi}{\partial r} - \frac{2g_o(\theta)}{r(\theta)} N \qquad (18)$$

Summary of Anomalies



Disturbing potential Φ

$$U = U_o + \Phi \qquad (19)$$

total	=	reference		disturbing
potential		potential		potential

Geoid height N

$$N = \frac{\Phi}{g_o(\theta)} \qquad (20)$$

Free-air gravity anomaly

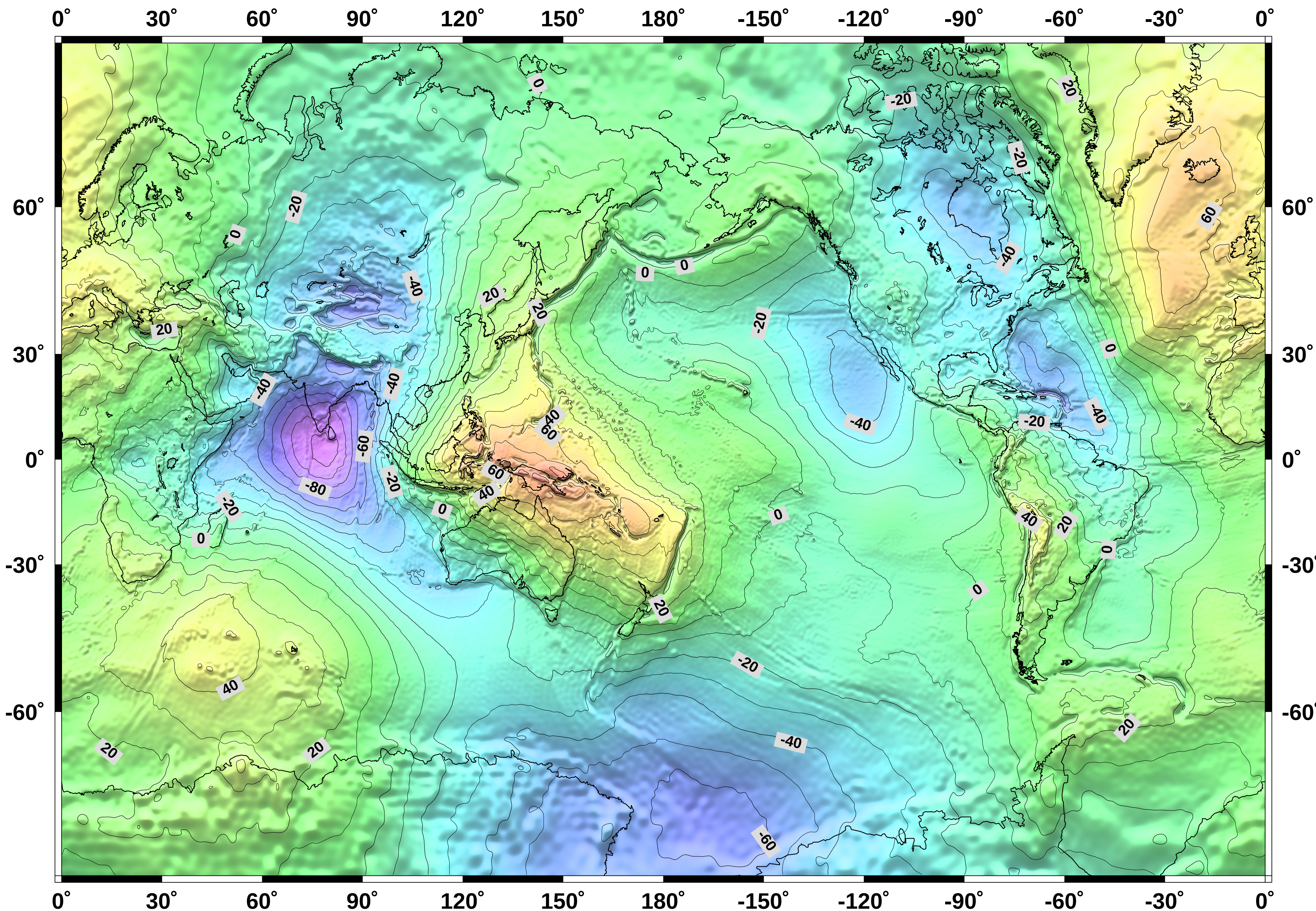
$$\Delta g = -\frac{\partial \Phi}{\partial r} - \frac{2g_o(\theta)}{r(\theta)} N \qquad (21)$$

Deflection of the vertical

The final type of anomaly, not yet discussed, is the *deflection of the vertical*. This is the angle between the normal to the geoid and the normal to the spheroid. There are two components north ξ and east, η.

$$\xi = -\frac{1}{a}\frac{\partial N}{\partial \theta} \qquad (22)$$

$$\eta = -\frac{1}{a\cos\theta}\frac{\partial N}{\partial \phi}$$



0° 30° 60° 90° 120° 150° 180° -150° -120° -90° -60° -30° 0°
60° 30° 0° -30° -60°
-80
-60
-40
-20
0
20
40
60